Narrow-band Frequency Modulation

KEEE343 Communication Theory

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Summary

- Narrowband Frequency Modulation
- Wideband Frequency Modulation

Narrow-Band Frequency Modulation

- Narrow-Band FM means that the FM modulated wave has narrow bandwidth.
- Consider the single-tone wave as a message signal:

$$m(t) = A_m \cos(2\pi f_m t)$$

- FM signal
  - Instantaneous frequency

$$\begin{aligned} f_i(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned} \qquad \Delta f = k_f A_m$$

  - Phase

$$\begin{aligned} \theta_i(t) &= 2\pi \int_0^t f_i(\tau)\, d\tau = 2\pi \left[f_c t + \frac{\Delta f}{2\pi f_m} \sin(2\pi f_m t) \right] \\ &= 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t) \end{aligned}$$

- Definitions
  - Phase deviation of the FM wave $\beta = \dfrac{\Delta f}{f_m}$
  - Modulation index of the FM wave: f_m
- Then, FM wave is

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

$$s(t) = A_c \cos(2\pi f_c t) \cos(\beta \sin(2\pi f_m t)) - A_c \sin(2\pi f_c t) \sin(\beta \sin(2\pi f_m t))$$

  - To be narrow-band FM, β should be small.
  - For small β compared to 1 radian, we can rewrite

$$\cos[\beta \sin(2\pi f_m t)] \approx 1, \qquad \sin[\beta \sin(2\pi f_m t)] \approx \beta \sin(2\pi f_m t)$$

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$

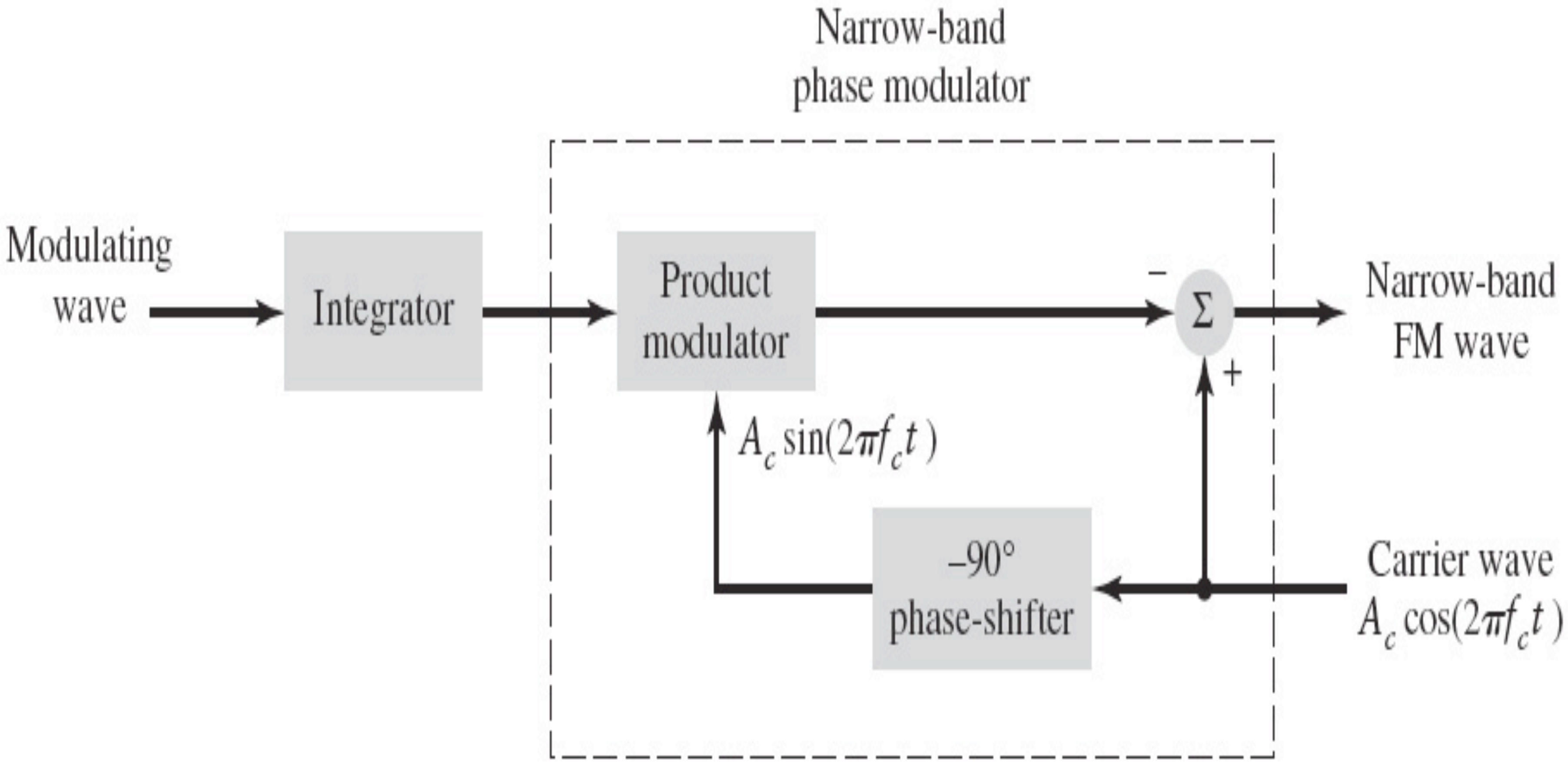

FIGURE 4.4 Block diagram of an indirect method for generating a narrow-band FM wave.

[Ref: Haykin & Moher, Textbook]

Polar Representation from Cartesian

- Consider the modulated signal

$$s(t) = a(t)\cos(2\pi f_c t + \phi(t))$$

which can be rewritten as

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

where

$$s_I(t) = a(t)\cos(\phi(t)), \quad \text{and}, \quad s_Q(t) = a(t)\sin(\phi(t))$$

and

$$a(t) = \left[s_I^2(t) + s_Q^2(t)\right]^{\frac{1}{2}}, \qquad \text{and} \qquad \phi(t) = \tan^{-1}\left[\frac{s_Q(t)}{s_I(t)}\right]$$

- Approximated narrow-band FM signal can be written as

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$

- Envelope

$$a(t) = A_c \left[1 + \beta^2 \sin^2(2\pi f_m t)\right] \approx A_c \left(1 + \frac{1}{2}\beta^2 \sin^2(2\pi f_m t)\right)$$

for small β

- Maximum value of the envelope

$$A_{\max} = A_c \left(1 + \frac{1}{2}\beta^2\right)$$

- Minimum value of the envelope

$$A_{\min} = A_c$$

- The ratio of the maximum to minimum value

$$\frac{A_{\text{max}}}{A_{\text{min}}} = \left(1 + \frac{1}{2}\beta^2\right)$$

- Rewriting the approximated narrow-band FM wave is

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta \cos(2\pi(f_c + f_m)t) - \frac{1}{2}\beta A_c \cos(2\pi(f_c - f_m)t)$$

- Average power

$$P_{\text{av}} = \frac{1}{2}A_c^2 + \left(\frac{1}{2}\beta A_c\right)^2 + \left(\frac{1}{2}\beta A_c\right)^2 = \frac{1}{2}A_c^2(1 + \beta^2)$$

- Average power of the unmodulated wave

$$P_c = \frac{1}{2}A_c^2$$

- Ration of the average power to the power of the unmodulated wave

$$\frac{P_{\mathrm{av}}}{P_c} = 1 + \beta^2$$

- Comparison with the AM signal

  - Approximated Narrow-band FM signal

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta \cos(2\pi(f_c + f_m)t) - \frac{1}{2}\beta A_c \cos(2\pi(f_c - f_m)t)$$

  - AM signal

$$s_{\mathrm{AM}}(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\mu A_c \left\{\cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]\right\}$$

difference between the narrow-band FM and AM waves

- Bandwidth of the narrow-band FM: $2f_m$

- Phasor interpretation

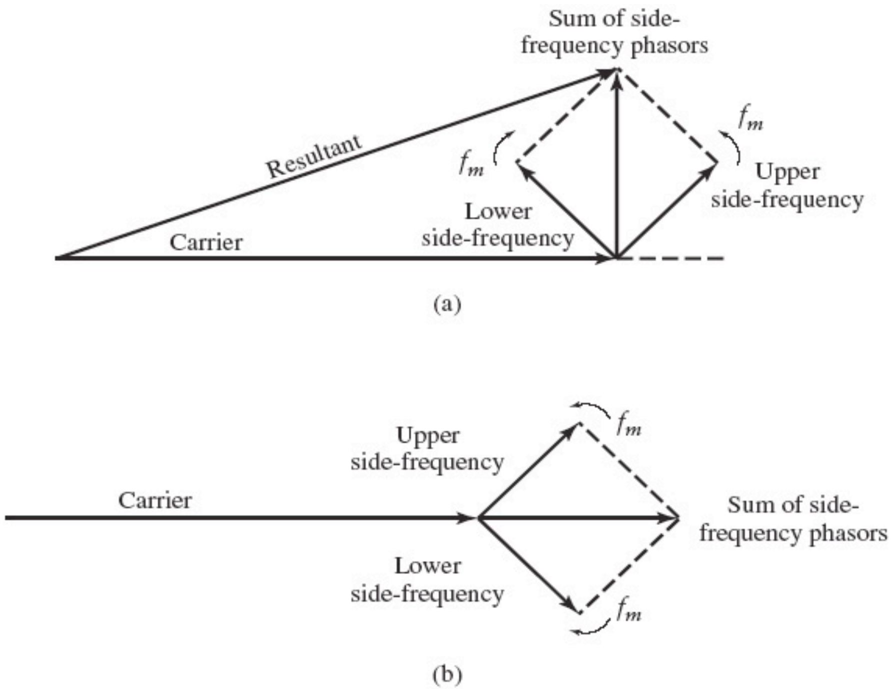

FIGURE 4.5 Phasor comparison of narrow-band FM and AM waves for sinusoidal modulation. (*a*) Narrow-band FM wave. (*b*) AM wave.

[Ref: Haykin & Moher, Textbook]

- Angle

$$\theta(t) = 2\pi f_c t + \phi(t) = 2\pi f_c t + \tan^{-1}(\beta \sin(2\pi f_m t))$$

- Using the power series of the tangent function such as

$$\tan^{-1}(x) \approx x - \frac{1}{3}x^3 + \cdots$$

- Angle can be approximated as

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{1}{3}\beta^3 \sin^3(2\pi f_m t)$$

- Ideally, we should have

$$\theta(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t)$$

- The harmonic distortion value is

$$D(t) = \frac{\beta^3}{3} \sin^3(2\pi f_m t)$$

- The maximum absolute value of D(t) is

$$D_{\max} = \frac{\beta^3}{3}$$

- For example for $\beta = 0.3,$

$$D_{\max} = \frac{0.3^3}{3} = 0.009 \approx 1\%$$

which is small enough for it to be ignored in practice.

Amplitude Distortion of Narrow-band FM

- Ideally, FM wave has a constant envelope
  - But, the modulated wave produced by the narrow-band FM differ from this ideal condition in two fundamental respects:
    - The envelope contains a residual amplitude modulation that varies with time
    - The angle $\theta_i(t)$ contains harmonic distortion in the form of third- and higher order harmonics of the modulation frequency f_m

Wide-Band Frequency Modulation

- Spectral analysis of the wide-band FM wave

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

or

$$s(t) = \Re\left[A_c \exp[j2\pi f_c t + j\beta \sin(2\pi f_m t)]\right] = \Re[\tilde{s}(t) \exp(j2\pi f_c t)]$$

where $\tilde{s}(t) = A_c \exp\left[j\beta \sin(2\pi f_m t)\right]$ is called "complex envelope".

Note that the complex envelope is a periodic function of time with a fundamental frequency f_m which means

$$\tilde{s}(t) = \tilde{s}(t + kT_m) = \tilde{s}(t + \frac{k}{f_m})$$

where $T_m = 1/f_m$

- Then we can rewrite

$$\begin{aligned}\tilde{s}(t) &= \tilde{s}(t + k/f_m) \\ &= A_c \exp[j\beta \sin(2\pi f_m (t + k/f_m))] \\ &= A_c \exp[j\beta \sin(2\pi f_m t + 2k\pi)] \\ &= A_c \exp[j\beta \sin(2\pi f_m t)]\end{aligned}$$

- Fourier series form

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t)$$

where

$$\begin{aligned}c_n &= f_m \int_{-1/(2f_m)}^{1/(2f_m)} \tilde{s}(t) \exp(-j2\pi n f_m t)\, dt \\ &= f_m A_c \int_{-1/(2f_m)}^{1/(2f_m)} \exp[j\beta \sin(2\pi f_m t) - j2\pi n f_m t]\, dt\end{aligned}$$

- Define the new variable: $x = 2\pi f_m t$

  Then we can rewrite

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx$$

- nth order Bessel function of the first kind and argument β

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]\, dx$$

- Accordingly

$$c_n = A_c J_n(\beta)$$

  which gives

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j2\pi n f_m t)$$

- Then the FM wave can be written as

$$\begin{aligned} s(t) &= \Re[\tilde{s}(t)\exp(j2\pi f_c t)] \\ &= \Re\left[A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\exp[j2\pi n(f_c + f_m)t]\right] \\ &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + n f_m)t] \end{aligned}$$

- Fourier transform

$$S(f) = \frac{A_c}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)\right]$$

which shows that the spectrum consists of an infinite number of delta functions spaced at $f = f_c \pm n f_m$ for $n = 0, +1, +2, ...$

Properties of Single-Tone FM for Arbitrary Modulation Index β

1. For different values of n

$$\begin{aligned} J_n(\beta) &= J_{-n}(\beta), \qquad &\text{for } n \text{ even} \\ J_n(\beta) &= -J_{-n}(\beta), \qquad &\text{for } n \text{ odd} \end{aligned}$$

2. For small value of β

$$\begin{aligned} J_0(\beta) &\approx 1, \\ J_1(\beta) &\approx \frac{\beta}{2} \\ J_n(\beta) &\approx 0, \qquad n > 2 \end{aligned}$$

6. The equality holds exactly for arbitrary β

$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

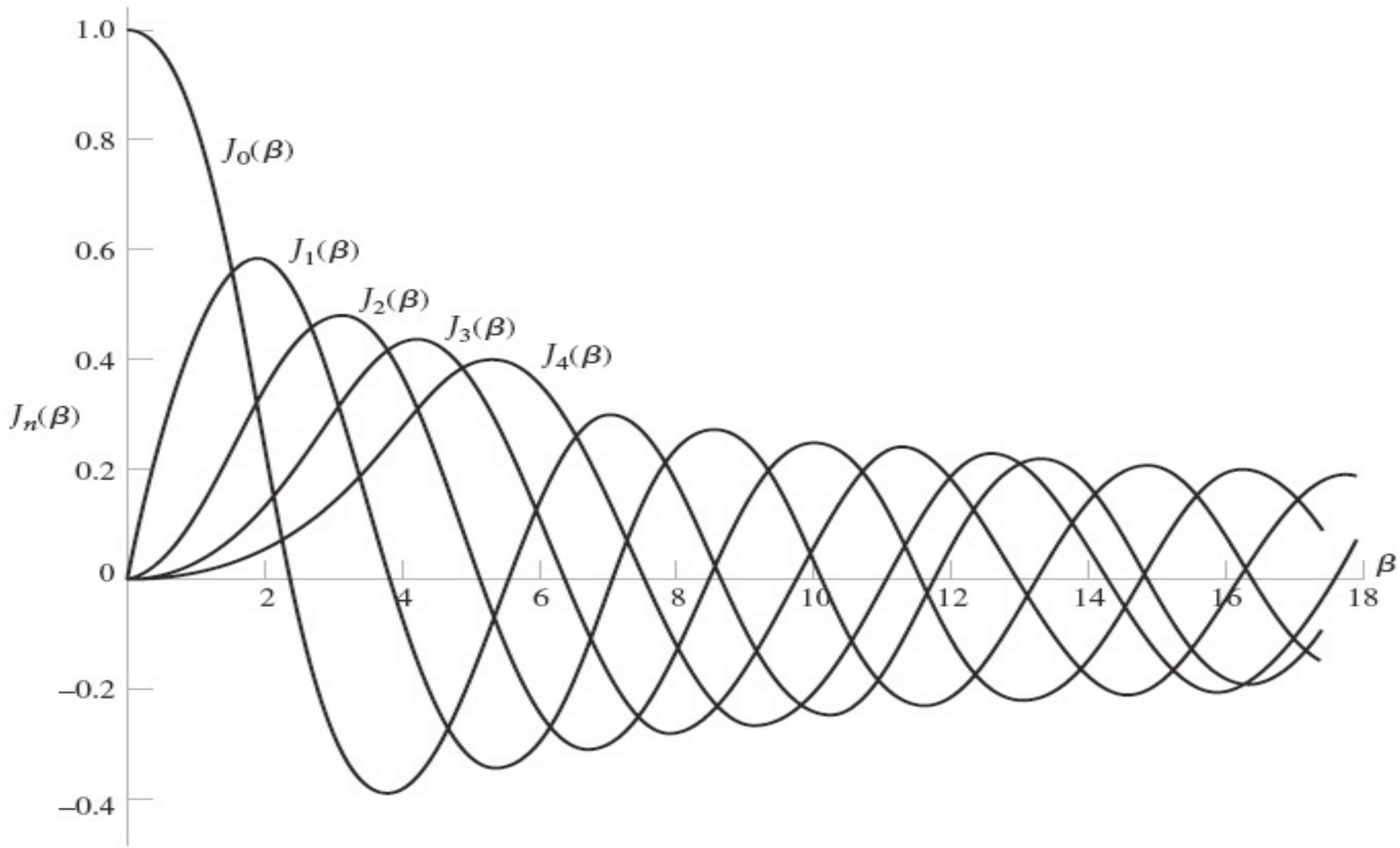

FIGURE 4.6 Plots of the Bessel function of the first kind, $J_n(\beta)$, for varying order n.

[Ref: Haykin & Moher, Textbook]

1. The spectrum of an FM wave contains a carrier component and and an infinite set of side frequencies located symmetrically on either side of the carrier at frequency separations of $f_m,\ 2f_m,\ 3f_m$...

2. The FM wave is effectively composed of a carrier and a single pair of side-frequencies at $f_c \pm f_m$.

3. The amplitude of the carrier component of an FM wave is dependent on the modulation index β .The averate power of such as signal developed across a 1-ohm resistor is also constant:

$$P_{\mathrm{av}} = \frac{1}{2} A_c^2$$

The average power of an FM wave may also be determined from

$$P_{\mathrm{av}} = \frac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta)$$