Discrete Mathematics: Lecture 15: Counting
counting

- combinatorics: the study of the number of ways to put things together into various combinations

- basic counting principles
  - product rule
  - sum rule
  - subtraction rule (inclusion-exclusion)
  - division rule
- pigeonhole principle
- permutation
- combination
basic counting principles: \textit{product rule}

If there are \( n_1 \) ways to do the first task and there are \( n_2 \) ways to do the second task, then there are \( n_1 n_2 \) ways to do the procedure of the first task and the second task in order.

A company with two employees, A and B, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

\[ 12 \times 11 = 132 \]
basic counting principles: product rule

How many different bit strings of length seven are there?

\[ 2^7 = 128 \]

How many functions are there from a set with \( m \) elements to a set with \( n \) elements?

\[ n \times n \times n \times \ldots n = n^m \]

How many one-to-one functions from a set with \( m \) elements to a set with \( n \) elements?

\[ n \times (n-1) \times (n-2) \times (n-m+1) \]
basic counting principles: sum rule

if a task can be done either in one of \( n_1 \) ways or in one of \( n_2 \) ways, where none of the set of \( n_1 \) ways is the same as any of the set of \( n_2 \) ways, then there are \( n_1 + n_2 \) ways to do the task.

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

\[ 23 + 15 + 19 = 57 \]
Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

\[
P = P_6 + P_7 + P_8
\]

\[
P_6 = (26+10)^6 - 26^6
\]

\[
P_7 = 36^7 - 26^7
\]

\[
P_8 = 36^8 - 26^8
\]
basic counting principles: **subtraction rule (inclusion-exclusion)**

If a task can be done **either in** $n_1$ **ways or** $n_2$ **ways**, then the number of ways to do the task is $n_1 + n_2 - \text{(the number of ways to do the task that are common to the two different ways)}$.

How many bit string of length eight start with a 1 bit or end with the two bits 00?

A bit string that begins with 1: $2^7 = 128$

A bit string that ends with 00: $2^6 = 64$

A bit string that begins with 1 and ends with 00: $2^5 = 32$

$128 + 64 - 32 = 160$
Basic counting principles: **division rule**

There are $n/d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, exactly $d$ of the $n$ ways correspond to way $w$.

How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

$$(4 \cdot 3 \cdot 2 \cdot 1) / 4 = 6$$
pigeonhole principle

if $k + 1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.

assigning 13 pigeons in 12 pigeonholes.
pigeonhole principle

In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

101 possible scores among 102 students, there must be at least 2 students with the same score.
generalized pigeonhole principle

if $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

proof by contradiction

suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects.

$$k(\lceil N/k \rceil - 1) < k((N/k + 1) - 1) = N$$

This is a contradiction because there are a total of $N$ objects.
What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

\[ \left\lfloor \frac{N}{5} \right\rfloor = 6 \]

\[ \frac{N - 1}{5} = 5 \quad N = 5 \cdot 5 + 1 = 26 \]

What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits from the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit

\[ \left\lfloor \frac{25,000,000}{8 \cdot 1,000,000} \right\rfloor = 4 \]

4 area codes are required
During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

\( a_j \): the number of games played on or before the \( j \)th day of the month

\( 1 \leq a_j \leq 45, \quad 15 \leq a_j + 14 \leq 59 \)

60 positive integers \( a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14 \leq 59 \)

\( \therefore \) two of these integers are equal.

\( a_i = a_j + 14 \)
permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects.

If \( n \) is a positive integer and \( r \) is an integer with \( 1 \leq r \leq n \), then there are

\[
P(n, r) = n(n-1)(n-2) \ldots (n-(r-1)) = \frac{n!}{(n-r)!}
\]

\( r \)-permutations of a set with \( n \) distinct elements.

\( S = \{a, b, c\} \)

2-permutation of \( S \): a,b; a,c; b,a; b,c; c,a; c,b

\[
P(3, 2) = 3 \cdot 2 = 6
\]
permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

\[ P(100, 3) = 100 \cdot 99 \cdot 98 = 970,200 \]

How many permutations of the letters ABCDEFGH contain the string ABC?

ABC, D, E, F, G, H

6!
A combination is finding the number of subsets of a particular size of a set with n elements.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$,

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{r! (n-r)!} = C(n, n-r)$$

$S = \{a, b, c, d\}$

2-combination of S: $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$

$C(4, 2) = \frac{4!}{(2!2!)} = 6$
How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

\[ C(10, 5) = \frac{10!}{5! \cdot 5!} = 252 \]

there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics and 4 from the computer science?

\[ C(9, 3) \cdot C(11, 4) = \frac{9!}{3! \cdot 6!} \cdot \frac{11!}{4! \cdot 7!} = 84 \cdot 330 = 27,720 \]
show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day?
A drawer contains 12 brown socks and 12 black socks. A boy takes socks out at random.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

b) How many socks must he take out to be sure that he has at least two black socks?
permutations vs. combination

A horse race will include 10 horses. You are given 2 betting options. You can chose the top three horses regardless of finishing order or you can chose the top three horses in exact order of finish. How many possibilities are there for each outcome?

Top 3 in any order: \( C(10,3) = \frac{n!}{r!(n-r)!} = 120 \)

Top 3 in exact order: \( P(10,3) = \frac{n!}{(n-r)!} = 720 \)
permutations with repetition

How many strings of length $r$ can be formed from the uppercase letters of the English alphabet?

$26^r = n^r$
Pick 3 pieces of a fruit from 2 boxes with apples and oranges. How many possible cases do we have?
combinations with repetition

How many ways are there to select 5 bills from a cash box containing $1 bills, $2 bills, $5 bills, $10 bills, $20 bills, $50 bills, and $100 bills? Assume that the order of the bills chosen does not matter and that there are at least 5 bills of each type.

\[ C(7-1+5, 5) \]

\[ C(n-1+r, r) = C(n-1+r, n-1) \]
How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where $x_1, x_2,$ and $x_3$ are nonnegative integers?

$C(3-1+11, 11) = 78$

How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where $x_1 \geq 1, x_2 \geq 2,$ and $x_3 \geq 3$ are nonnegative integers?

$C(3-1+5, 5) = C(7, 5) = 21$
## Combination and Permutation with or without Repetition

<table>
<thead>
<tr>
<th>Type</th>
<th>Repetition</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>r-permutation P(n,r)</td>
<td>no</td>
<td>( \frac{n!}{(n-r)!} )</td>
</tr>
<tr>
<td>r-combination C(n,r)</td>
<td>no</td>
<td>( \frac{n!}{r!(n-r)!} )</td>
</tr>
<tr>
<td>r-permutation</td>
<td>yes</td>
<td>( n^r )</td>
</tr>
<tr>
<td>r-combination</td>
<td>yes</td>
<td>( \frac{(n+r-1)!}{r!(n-1)!} )</td>
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The number of different permutations of $n$ objects, where there are $n_1$ indistinguishable objects of type 1, $n_2$ indistinguishable objects of type 2, … and $n_k$ indistinguishable objects of type $k$ is

$$\frac{n!}{n_1!n_2!...n_k!}$$

How many different strings can be made by reordering the letters of the word SUCCESS?

$$\frac{7!}{3!2!}$$
distributing objects into boxes

distinguishable objects into distinguishable boxes

\[
\frac{n!}{n_1!n_2!...n_k!}
\]

how many ways are there to distribute 5 cards to 4 players from the different 52 cards?

\[
\binom{52}{5} \binom{47}{5} \binom{42}{5} \binom{37}{5} = \frac{52!}{5!5!5!5!32!}
\]
distributing objects into boxes

indistinguishable objects \( r \) into distinguishable boxes \( n \)

\[ C(n - 1 + r, \ n - 1) \]

How many ways are there to place 10 indistinguishable balls into 8 different bins?

\[ C(8 - 1 + 10, 8) = 19448 \]
distributing objects into boxes

distinguishable objects into indistinguishable boxes

How many ways are there to put 4 different employees into 3 indistinguishable offices when each office can contain any number of employees?

$S(n, j)$: the number of ways to distribute $n$ distinguishable objects into $j$ indistinguishable boxes

$S(4,1) = 1$

$S(4,2) = C(4, 1) + C(4, 2)/2 = 4 + 3 = 7$

$S(4,3) = C(4, 2) = 6$
distributing objects into boxes

indistinguishable objects into indistinguishable boxes

How many ways are there to pack 6 copies of the same book into 4 identical boxes, where a box can contain as many as six books?

6
5, 1
4, 2
4, 1, 1
3, 3
3, 2, 1
3, 1, 1, 1
2, 2, 2
2, 2, 1, 1
How many ways can 13 books be placed on 3 distinguishable shelves

a) if the books are indistinguishable copies of the same title?

b) if no two books are the same, and the positions of the books on the shelves matter?