Chapter 11: RISK NEUTRAL TREES AND DERIVATIVE PRICING

11.1 RISK NEUTRAL TREES

- **11.1.1 The Ho-Lee Model**
- **11.1.2 The Simple Black, Derman and Toy (BDT) Model**
- **11.1.3 Comparison of the Two Models**
- **11.1.4 Risk Neutral Trees and Interest Rates**

11.1.1 THE HO-LEE MODEL

- The Ho-Lee model is one of the simplest models that exactly fits the term structure of interest rates
- The model is specified as follows: let $r_{i,j}$ be the continuously compounded interest rate in node j between steps i and $i + 1$:

  $$r_{i+1,j} \quad = \quad r_{i,j} + \theta_i \times \Delta + \sigma \times (\Delta)^{1/2}$$
  $$r_{i+1,j+1} \quad = \quad r_{i,j} + \theta_i \times \Delta - \sigma \times (\Delta)^{1/2}$$

  with R.N. probability $p^* = ½$
- Recall that on multistep trees we denote:
- $P_{i,j}(k)$ = Bond price at time i in node j with maturity at (step) k

11.1.1 THE HO-LEE MODEL

- An example:
  - Consider the term structure of interest rates on January 8, 2002: the zero coupon bond expiring on date $k = 1$ is $P_0(1) = 99.1338$, implying $r_0 = 1.74\%$ (the root of the tree)
  - In the data, the zero coupon bond expiring on date $k = 2$ is $P_0(2) = 97.8925$
  - We now choose θ_0 so that the binomial tree exactly gives $P_0(2)$ as price

$$r_{1,0} = 1.74\% + \theta_0 \times \Delta + \sigma(\Delta)^{½} \text{ with RN probability } p^* = ½$$
$$r_{1,1} = 1.74\% + \theta_0 \times \Delta - \sigma(\Delta)^{½} \text{ with RN probability } p^* = ½$$

at the time the data gave $\sigma = 0.0173$

  - We can now choose θ_0 so that the following equation is satisfied

$$97.8925 = e^{-r_0 \times \Delta} \times (0.5 \times e^{-r_{1,0} \times \Delta} + 0.5 \times e^{-r_{1,1} \times \Delta}) \times 100$$

Price of zero in the data = Risk neutral price from binomial tree

  - Given $r_0 = 1.74\%$ and $\sigma = 0.0173$, $r_{1,0}$ and $r_{1,1}$ depend only on the level of θ_0
  - Thus, we have one equation with one unknown
  - Using a search algorithm, we find $\theta_0 = 1.5674\%$
  - Given this value for θ_0, the two interest rates are $r_{1,0} = 3.75\%$ and $r_{1,1} = 1.30\%$.

11.1.1 THE HO-LEE MODEL

- An example (cont'd):
  - In the data, the zero coupon expiring on date $k = 3$ has price $P_0(3) = 96.1462$
  - Keeping θ_0 as determined in the previous step, we now look for θ_1 such that the tree exactly yields a price $P_0(3) = 96.1462$
  - Rather than using an equation to find θ_1, we use the binomial tree itself; specifically, let us set up a three-step binomial tree for a given θ_1, e.g. $\theta_1 = 0$
  - This tree will provide a bond value different from the one that we need
  - However, we can then vary θ_1 until we reach the correct value for the bond
  - Table 11.1 shows the result:
    - On the left-hand side there is an interest rate tree and bond price for the case in which $\theta_1 = 0$
    - On the right-hand side of the table, instead, there is the interest rate tree and the bond price for the θ_1 that exactly matches the bond price in the data for maturity $k = 3$
  - As can be seen comparing the two trees, the one on the right-hand side has nodes $r_{2,0}$, $r_{2,1}$ and $r_{2,2}$ that are higher than the corresponding nodes on the left-hand side
  - θ_1 had to be chosen greater than 0 to match the term structure of interest rates

Table 11.1 Two Trees for a Zero Coupon Bond Expiring on $k = 3$

Price to Match: 96.1462

$\theta_1 = 0$

Interest Rate Tree

1.74%	3.75%	4.97%
	1.30%	2.52%
		0.08%

Zero Coupon Bond Price

96.6722	96.3241	97.5455	100
	98.7098	98.7461	100
		99.9614	100
			100

Optimal $\theta_1 = 0.021824$

Interest Rate Tree

1.74%	3.75%	6.06%
	1.30%	3.61%
		1.17%

Zero Coupon Bond Price

96.1462	95.8000	97.0147	100
	98.1727	98.2088	100
		99.4175	100
			100

11.1.1 THE HO-LEE MODEL (cont.)

Table 11.2 The Risk Neutral Ho-Lee Interest Rate Tree

Time T	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Period i	0	1	2	3	4	5	6	7	8	9	10
$\theta_i(\times 100)$	1.5675	2.1824	1.4374	1.7324	0.7873	0.0423	-0.0628	0.4322	0.9271	0.1202	
j											
0	1.74	3.75	6.06	8.00	10.09	11.71	12.95	14.15	15.59	17.27	18.56
1		1.30	3.61	5.56	7.65	9.26	10.51	11.70	13.14	14.83	16.11
2			1.17	3.11	5.20	6.82	8.06	9.25	10.69	12.38	13.66
3				0.66	2.75	4.37	5.61	6.81	8.25	9.93	11.22
4					0.31	1.92	3.17	4.36	5.80	7.49	8.77
5						-0.52	0.72	1.91	3.35	5.04	6.32
6							-1.73	-0.53	0.91	2.59	3.88
7								-2.98	-1.54	0.15	1.43
8									-3.99	-2.30	-1.02
9										-4.75	-3.46
10											-5.91

11.1.2 The Simple Black, Derman and Toy (BDT) Model

- The main drawback of the Ho-Lee model is that it allows negative interest rates
- The BDT solves this by defining:

$$z_{i,j} = \ln(r_{i,j})$$

- Then for *zi,j* we have the process:

$$z_{i+1,j} = z_{i,j} + \theta_i \times \Delta + \sigma \times (\Delta)^{1/2}$$

$$z_{i+1,j+1} = z_{i,j} + \theta_i \times \Delta - \sigma \times (\Delta)^{1/2}$$

with R.N. probability $p^* = 1/2$

- While $z_{i,j}$ can be negative, $r_{i,j}$ is always positive

11.1.2 The Simple Black, Derman and Toy (BDT) Model

- An example:
  - The strategy to fit the term structure of interest rates is the same as for that of the Ho-Lee model
    - First look for θ_0 that yields exactly the price of a bond maturing on $k = 2$
    - Then, we move to find θ_1 that fits exactly the price of the bond maturing on $k = 3$
    - And so on
  - Table 11.3 shows the risk neutral tree
  - The important detail to notice is the level of σ that we need to choose for the model
  - Note that differently from the Ho-Lee model, now σ is the volatility of log-interest rates $z_i = \log(r_i)$
  - As such, it must be estimated from a log interest rate series
  - Taking log differences in monthly interest rates from 1961/12 to 2001/12, we obtain an (annualized) level of volatility equal to σ = 21.42%.

11.1.2 The Simple Black, Derman and Toy (BDT) Model (cont.)

Table 11.3 The Risk Neutral Simple Black, Derman, and Toy Interest Rate Tree

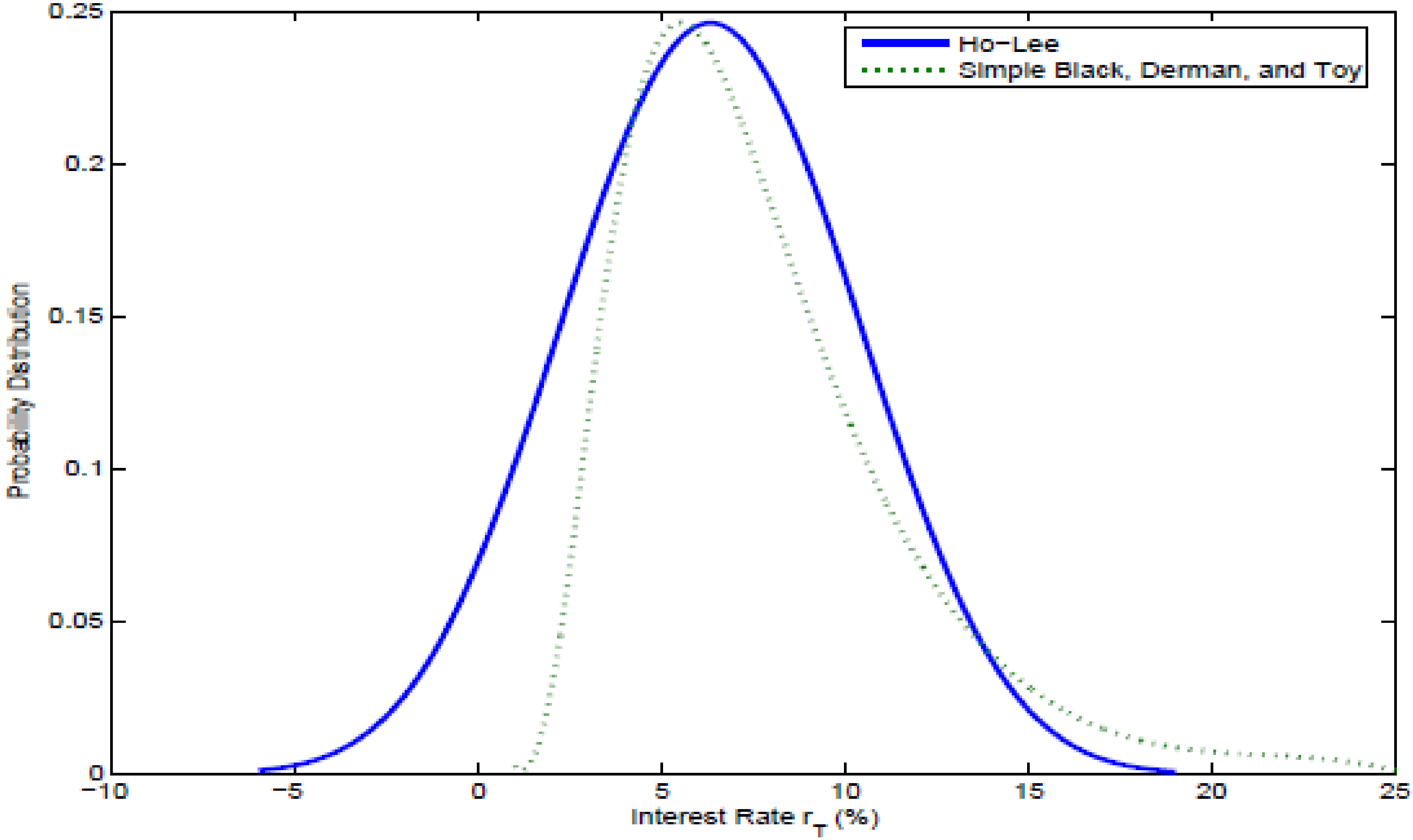
time T	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
period i	0	1	2	3	4	5	6	7	8	9	10
$\theta_i(\times 100)$	71.82	69.16	33.48	33.79	11.82	-2.30	-4.38	4.55	12.81	-1.26	
j											
0	1.74	2.90	4.77	6.56	9.03	11.15	12.83	14.60	17.38	21.56	24.93
1		2.14	3.52	4.84	6.67	8.24	9.47	10.78	12.84	15.92	18.41
2			2.60	3.58	4.93	6.08	7.00	7.97	9.48	11.76	13.60
3				2.64	3.64	4.49	5.17	5.88	7.00	8.69	10.05
4					2.69	3.32	3.82	4.35	5.17	6.42	7.42
5						2.45	2.82	3.21	3.82	4.74	5.48
6							2.08	2.37	2.82	3.50	4.05
7								1.75	2.09	2.59	2.99
8									1.54	1.91	2.21
9										1.41	1.63
10											1.21

11.1.3 Comparison of the Two Models

- By construction, the two models are equally able to fit the term structure of interest rates
- However, the two models generate important differences in the implied risk neutral probability distribution of interest rates in the future
  - The Ho-Lee model gives non-zero probability to negative interest rates, and small probability to high interest rates
  - The Simple BDT model gives essentially zero probability to interest rates below 1%, but assigns higher probability to high interest rates
- These differences are not important for bond prices, as both models exactly match the term structure of interest rates
- However, they will generate important differences for other securities that have asymmetric payoff structures, such as options

11.1.3 Comparison of the Two Models (cont.)

Figure 11.1 The Risk Neutral Distribution of Interest Rates at $T = 5$



11.1.3 Comparison of the Two Models (cont.)

- Consider a structured bond with payoff:

$$\max(11 \times 100 \times r_T, 94)$$

- Under each model specified we get:
- Price under Ho-Lee: $80.0645
- Price under BDT: $78.9135
- The lower price in the Simple Black, Derman and Toy model highlights the differences of the model:
  - Although the positive skewness of the risk neutral distribution in the BDT model implies a higher risk neutral expected payoff for the Simple BDT model, the higher interest rates implied by the model also imply a higher discount applied to the payoff.
  - The higher discount effect more than compensates for the higher expected return

11.1.4 Risk Neutral Trees and Future Interest Rates

- There is often a temptation to interpret too much from the implied risk neutral interest rate trees
- Remember that a risk neutral interest rate tree's only purpose is to compute the price of interest rate securities through no arbitrage
- This has little to do with the real world expectation of future interest rates
- It should be noted that:
  - The BDT model does not allow enough (risk neutral) probability mass to low interest rates, which makes this model underperform in low interest rate environments
  - In contrast, the Ho-Lee model allows perhaps too much (risk neutral) probability to low interest rates, and in fact even to negative interest rates
- Derivative security prices are very sensitive to this distributional differences

11.2 USING RISK NEUTRAL TREES

- **11.2.1 Intermediate Cash Flows**
- **11.2.2 Caps and Floors**
- **11.2.3 Swaps**
- **11.2.4 Swaptions**

11.2.1 Intermediate Cash Flows

- Given a tree, we can insert any type of known cash flow:

  $$P_{i,j} = e^{-r_{i,j} \times \Delta} \times \left(\tfrac{1}{2} P_{i+1,j} + \tfrac{1}{2} P_{i+1,j+1} + CF(i+1)\right)$$

  where $CF(i+1)$ is the cash flow paid at time $i+1$

11.2.1 Intermediate Cash Flows

- An example:
  - Consider the price of a 1.5-year, 3% coupon bond on January 8, 2002
  - use the Simple Black, Derman, and Toy interest rate model
  - We calculate the price of the coupon bond using the tree in Table 11.4
  - In each step, we add the cash flow $CF(i+1) = 1.5$ (= 3% × 100/2) to the price in the following period, and take the present value
  - So, for example, the value of the bond if the interest rate goes up twice (to $r_{2,uu} = 4.77\%$) is equal to the present value of the bond value in the next period, equal to \$100, plus the coupon to be received next period, equal to \$1.5
  - The present value is then $P_{2,uu} = \$99.1094$
  - The prices on the tree are ex-coupon prices, that is, the price of the bond right after the coupon has been paid

11.2.2 Caps and Floors

- A plain vanilla cap with maturity T, strike rate r_K, and notional N is a security that pays a stream of cash flows at given dates $T_1, T_2, \ldots, T_m = T$, according to the formula:

  $$CF(T_i) = \Delta \times N \times \max(r_n(T_i - \Delta) - r_K, 0)$$

  where n is the number of payments per year and $\Delta = 1/n = T_i - T_{i-1}$ is the amount of time between payments and $r_n(T)$ is a reference floating rate with compounding frequency n (e.g. the 6-month T-bill rate or LIBOR)
- A plain vanilla floor pays cash flows according to:

  $$CF(T_i) = \Delta \times N \times \max(r_K - r_n(T_i - \Delta), 0)$$
- It is important to note that cash flows occurring at T_i are determined at the previous node $T_i - \Delta$
- Caps and floors are easily priced through trees

11.2.2 Caps and Floors (cont.)

- Consider a cap where cash flow is determined by:

$$CF(T_i) = \Delta \times N \times \max(r_n(i,j) - r_K, 0)$$

  where:

$$r_n(i,j) = n \times (e^{r_{i,j}} \times \Delta - 1)$$

  is corresponding interest rate with compounding frequency n
- Given these cash flows, we use the backward recursive formula to obtain the value of the cap along the tree
- $V_{i,j}$ = Value at time/node (i,j) of cash flows at times $k > i$
- $V_{i,j} = e^{-r_{i,j} \times \Delta} \times (½\, V_{i+1,j} + ½\, V_{i+1,j+1} + CF_{i,j}(i+1))$

11.2.2 Caps and Floors (cont.)

- An example:
- Consider the value on January 8, 2002, of a 1.5-years cap, with semi-annual payment ($n = 2$, $\Delta = 0.5$), and with strike rate $r_K = 3\%$ (notional $N = 100$); we apply the Simple Black, Derman, and Toy risk neutral tree
- We proceed in two steps:
  - **Cash Flow Tree:** The first step to obtain the price of the cap is to build a cash flow tree, that is, a tree that defines the cash flow that is determined (not paid) in a given node (i,j) (see tree in Table 11.5)

    The cash flow tree in this table also shows not only the time of the formation of the cash flows, but also when they would be paid (i.e., one period later)

    The corresponding semi-annually compounded interest rate is:

    $$r_2(2,uu) = 2 \times (e^{4.77\% / 2} - 1) = 4.82\%$$

    Thus, the cash flow determined at time/node ($2,uu$) is:

    $$C_{2,uu}(3) = 100 / 2 \times \max(4.82\% - 2.5\%, 0) = 1.162$$

    Note, however, this cash flow is not paid at time ($2,uu$) but at time $i = 3$, as the tree shows
  - **Cap Value Tree:** Given the cash flow tree, we can compute the value of the cap by using the backward formula

    The resulting tree is in Table 11.6; we obtain a value of the cap at time $i = 0$: $V_0 = \$0.647$

Table 11.5 The Cash Flow Tree of a 1.5-Year Cap

$i = 0$ $t = 0$	$i = 1$ $t = 0.5$	$i = 2$ $t = 1$	$i = 3$ $t = 1.5$
		$r_{2,uu} = 4.77\%$ $r_2(2, uu) = 4.82\%$ $CF_{2,uu}(3) = 1.162$	$\longrightarrow$ paid here
	$r_{1,u} = 2.90\%$ $r_2(1, u) = 2.92\%$ $CF_{1,u}(2) = 0.210$	$\longrightarrow$ paid here	
$r_0 = 1.74\%$ $r_2(0) = 1.75\%$ $CF_0(1) = 0$		$r_{2,ud} = 3.52\%$ $r_2(2, ud) = 3.55\%$ $CF_{2,ud}(3) = 0.526$	$\longrightarrow$ paid here
	$r_{1,d} = 2.14\%$ $r_2(1, d) = 2.15\%$ $CF_{1,d}(2) = 0$		
		$r_{2,dd} = 2.60\%$ $r_2(2, dd) = 2.62\%$ $CF_{2,dd}(3) = 0.059$	$\longrightarrow$ paid here

Table 11.6 The 1.5-Year Cap Value Tree

$i = 0$ $t = 0$	$i = 1$ $t = 0.5$	$i = 2$ $t = 1$	$i = 3$ $t = 1.5$
		$V_{2,uu} =$ $= e^{4.77\%/2} \times 1.162 = 1.135$	$CF_{2,uu}(3) = 1.162$
	$V_{1,u} = e^{-2.90\%/2} \times$ $\times [\frac{1}{2}(1.135 + 0.517) + 0.210]$ $= 1.021$		
$V_0 = e^{-1.74\%/2} \times$ $\times [\frac{1}{2}(1.021 + 0.285) + 0]$ $= 0.647$		$V_{2,ud} =$ $= e^{-3.52\%/2} \times 0.526 = 0.517$	$CF_{2,ud}(3) = 0.526$
	$V_{1,d} = e^{-2.14\%/2} \times$ $\times [\frac{1}{2}(0.517 + 0.058) + 0]$ $= 0.285$		
		$V_{2,dd} =$ $= e^{-2.60\%/2} \times 0.059 = 0.058$	$CF_{2,dd}(3) = 0.058$

Table 11.7 A 5-Year Cap

Panel A: Cash Flow Tree

Period i / Node j	0	1	2	3	4	5	6	7	8	9
0	0.00	0.21	1.16	2.08	3.37	4.48	5.37	6.32	7.83	10.13
1		0.00	0.53	1.20	2.14	2.95	3.60	4.29	5.38	7.04
2			0.06	0.55	1.25	1.84	2.31	2.81	3.61	4.81
3				0.08	0.59	1.02	1.37	1.74	2.31	3.19
4					0.10	0.42	0.68	0.95	1.37	2.01
5						0.00	0.17	0.37	0.68	1.15
6							0.00	0.00	0.17	0.52
7								0.00	0.00	0.05
8									0.00	0.00
9										0.00

Panel B: Cap Value Tree

Period i / Node j	0	1	2	3	4	5	6	7	8	9
0	9.44	12.19	15.10	17.35	18.93	19.44	18.89	17.29	14.33	9.10
1		6.86	9.21	11.25	12.76	13.42	13.25	12.25	10.22	6.50
2			4.64	6.45	7.89	8.68	8.82	8.33	7.06	4.53
3				2.84	4.13	5.00	5.40	5.32	4.65	3.05
4					1.46	2.24	2.79	3.03	2.83	1.95
5						0.52	0.92	1.30	1.47	1.12
6							0.12	0.23	0.44	0.51
7								0.01	0.03	0.05
8									0.00	0.00
9										0.00

11.2.3 Swaps

- Valuation of swaps can be obtained from the discount factor, yet understanding the dynamics of the value of the interest rate swap on an interest rate tree is instrumental to obtaining the price of other interest rate derivatives
- The cash flow for a plain vanilla swap is:

$$CF(T_i) = \Delta \times N \times (r_n(T_i - \Delta) - c)$$

where c is the swap rate

11.2.3 Swaps (cont.)

- The methodology to value a swap is identical to the one used to value a cap
  - Compute the cash flow tree using
    $$CF_{i,j}(i+1) = \Delta \times N \times (r_n(i,j) - c)$$
    where recall
    $$r_n(i,j) = n \times (e^{r_{i,j}} \times \Delta - 1)$$
  - Compute the value of the swap on the tree as the present value of the risk neutral expectation of future cash flows by moving backward on the tree:

$$V_{i,j}(k,c) = e^{-r_{i,j} \times \Delta} \times (½\, V_{i+1,j}(k,c) + ½\, V_{i+1,j+1}(k,c) + CF_{i,j}(i+1))$$

where: $V_{i,j}(k,c)$ = Value of the swap in (i,j) with maturity (k) and swap rate c

11.2.3 Swaps (cont.)

- An example:
  - Consider a 5-year fixed-for-floating swap on January 8, 2002, defined on the 6-month T-bill rate and with semi-annual payments; the swap rate is:

$$c = \frac{1}{2}\frac{1 - Z(0,10)}{\sum_{i=1}^{10} Z(0,i)} = 4.49\%$$

  - Recall that this swap rate is the one that makes the value of the interest rate swap equal to zero at inception
  - Given the Simple Black, Derman and Toy, we obtain the cash flow tree and swap value tree in Panels A and B of Table 11.8, respectively
  - The comforting fact is that the root of the swap value tree, in Panel B, indeed gives $V_0 = 0$, as it should be from the definition of c
  - This is not surprising, as the tree used to value this swap was calibrated to zero coupon bonds
  - However, we have confirmation that the tree methodology works, as it correctly values the interest rate swap

Table 11.8 A 5-Year Swap Tree

Panel A: Cash Flow Tree

Period i / Node j	0	1	2	3	4	5	6	7	8	9
0	-1.37	-0.78	0.17	1.09	2.38	3.49	4.38	5.33	6.83	9.14
1		-1.17	-0.47	0.21	1.15	1.96	2.61	3.30	4.39	6.04
2			-0.93	-0.44	0.25	0.84	1.32	1.82	2.61	3.81
3				-0.91	-0.41	0.03	0.37	0.74	1.32	2.20
4					-0.89	-0.57	-0.32	-0.05	0.38	1.02
5						-1.01	-0.82	-0.63	-0.31	0.15
6							-1.20	-1.05	-0.82	-0.48
7								-1.36	-1.20	-0.94
8									-1.47	-1.28
9										-1.54

Panel B: Swap Value Tree

Period i / Node j	0	1	2	3	4	5	6	7	8	9
0	0.00	4.27	8.18	11.38	13.86	15.22	15.50	14.72	12.58	8.20
1		-1.53	2.04	5.04	7.48	9.03	9.71	9.58	8.42	5.58
2			-2.79	0.05	2.44	4.14	5.18	5.58	5.21	3.60
3				-3.83	-1.47	0.36	1.67	2.51	2.77	2.10
4					-4.47	-2.54	-1.00	0.18	0.92	0.99
5						-4.74	-3.02	-1.58	-0.46	0.15
6							-4.55	-2.90	-1.50	-0.47
7								-3.89	-2.27	-0.93
8									-2.85	-1.27
9										-1.52

11.2.4 Swaptions

- A swaption, or option on a swap, is an interest rate contract between two counterparties in which one counterparty (the option buyer) has the right, but not the obligation, to enter at a pre-specified time T into a given interest rate swap with maturity $T^{swap} > T$ and (strike) swap rate r_K
- The other counterparty (the option seller) has the obligation to take the other side of the swap contract if the option buyer exercises the option
- Two main types of plain vanilla swaptions are the following:
    - A receiver swaption is an option to enter into a swap and receive the fixed rate r_K
    - A payer swaption is an option to enter into a swap and pay the fixed rate r_K

11.2.4 Swaptions (cont.)

- An example:
  - How do we value a swaption?
    - Consider a European payer swaption with two years to maturity ($i = 4$), to enter at $i = 4$ into a 3-year swap and pay the fixed rate $r_K = 4.49\%$
    - The maturity date of the swap is then five years from now, i.e. $k = 10$

1. Compute the tree of the underlying swap value whose swap rate is equal to the swaption's strike rate r_K
2. Compute the swaption payoff at time $i = 4$:

$$\max(V_{4,j}(10, r_K), 0)$$

3. Use the risk neutral binomial tree to compute the price of the swaption from its payoff

Table 11.9 A 2-Year Payer Swaption

i / j	0	1	2	3	4
0	3.41	5.11	7.41	10.33	13.86
1		1.76	2.97	4.84	7.48
2			0.59	1.20	2.44
3				0.00	0.00
4					0.00

11.3 IMPLIED VOLATILITIES AND THE BLACK, DERMAND AND TOY MODEL

- **11.3.1 Flat and Forward Volatility**

- **11.3.2 Forward Volatility and the Black, Derman and Toy Model**

11.3 IMPLIED VOLATILITIES AND THE BLACK, DERMAND AND TOY MODEL

- The **empirical volatility** of interest rates is the level of interest rate variation σ computed from a time series of past interest rate changes

  For instance, in the two models discussed, the empirical volatility is:

  Ho-Lee model : σ = std.dev. of $(r_{t+\Delta} - r_t)$

  Simple BDT: σ = std.dev. of $(\ln(r_{t+\Delta}) - \ln(r_t))$
- Consider a given derivative security, such as a cap, with maturity T and strike rate r_K, and let $cap^{Data}(T, r_K)$ be the current price level of the cap

  The **implied volatility** of this cap is the level of interest rate variation σ such that the chosen interest rate model yields a price of the cap identical to the $cap^{Data}(T, r_K)$

11.3 IMPLIED VOLATILITIES AND THE BLACK, DERMAND AND TOY MODEL (cont.)

Table 11.10 Swap Rates and Cap Prices on November 1, 2004

Maturity T	Swap Rates $c(0,T)$(%)	Discount $Z(0,T)$	Cap Prices Data	Cap Prices Simple BDT Model	Cap Prices Ho Lee Model
0.25	2.1800	99.4580	–	–	–
0.50	2.3177	98.8510	0.0456	0.0400	0.0689
0.75	2.4420	98.1899	0.1059	0.0948	0.1512
1.00	2.5550	97.4834	0.1859	0.1520	0.2349
1.25	2.6586	96.7385	0.2887	0.2106	0.3366
1.50	2.7546	95.9598	0.4157	0.3038	0.4457
1.75	2.8451	95.1503	0.5662	0.3984	0.5670
2.00	2.9320	94.3109	0.7364	0.4982	0.7050
2.25	3.0167	93.4417	0.9201	0.6062	0.8485
2.50	3.0991	92.5456	1.1129	0.7229	1.0008
2.75	3.1784	91.6268	1.3126	0.8586	1.1579
3.00	3.2540	90.6899	1.5194	0.9961	1.3252
3.25	3.3254	89.7397	1.7352	1.1386	1.4911
3.50	3.3930	88.7778	1.9598	1.2838	1.6643
3.75	3.4577	87.8050	2.1916	1.4344	1.8415
4.00	3.5200	86.8212	2.4288	1.5889	2.0247
4.25	3.5805	85.8263	2.6691	1.7542	2.2129
4.50	3.6393	84.8218	2.9117	1.9208	2.4007
4.75	3.6962	83.8102	3.1562	2.0954	2.5946
5.00	3.7510	82.7938	3.4029	2.2706	2.7889

Original Data Source: Bloomberg.

11.3.1 Flat and Forward Volatility

- The Ho-Lee model appears to overprice short term caps, and underprice long term caps, while the Simple BDT model in this case always underprices
- One possible problem with the model is that the volatility σ has been mis-measured
  - The volatility of interest rates is time varying, and thus we may be using the wrong level of volatility
- A single value of σ that makes the observed cap price consistent with the model does not exist

11.3.1 Flat and Forward Volatility (cont)

Table 11.12 Cap Implied Volatilities: The Simple BDT and Ho-Lee Models

		SIMPLE BDT				HO-LEE			
		Implied Volatilities σ for Caps with Maturity T (in Parenthesis)							
		0.188	0.2291	0.30277	0.28504	0.00458	0.00584	0.01006	0.010997
Maturity	Data	$(T = 0.5)$	$(T = 1)$	$(T = 3)$	$(T = 5)$	$(T = 0.5)$	$(T = 1)$	$(T = 3)$	$(T = 5)$
0.50	0.0456	0.0456	0.0518	0.0628	0.0602	0.0456	0.0534	0.0796	0.0854
0.75	0.1059	0.1061	0.1184	0.1398	0.1347	0.1047	0.1204	0.1727	0.1843
1.00	0.1859	0.1674	0.1859	0.2262	0.2166	0.1644	0.1859	0.2723	0.2926
1.25	0.2887	0.2385	0.2715	0.3324	0.3178	0.2224	0.2601	0.3899	0.4187
1.50	0.4157	0.3374	0.3735	0.4542	0.4336	0.3105	0.3510	0.5128	0.5522
1.75	0.5662	0.4428	0.4958	0.5944	0.5710	0.4053	0.4544	0.6594	0.7094
2.00	0.7364	0.5583	0.6258	0.7492	0.7182	0.4988	0.5651	0.8076	0.8668
2.25	0.9201	0.6790	0.7648	0.9274	0.8884	0.5996	0.6811	0.9697	1.0353
2.50	1.1129	0.8154	0.9202	1.1118	1.0640	0.7077	0.8005	1.1441	1.2233
2.75	1.3126	0.9577	1.0819	1.3115	1.2559	0.8256	0.9297	1.3287	1.4213
3.00	1.5194	1.1156	1.2543	1.5194	1.4528	0.9508	1.0625	1.5194	1.6245
3.25	1.7352	1.2751	1.4373	1.7454	1.6709	1.0740	1.2044	1.7071	1.8265
3.50	1.9598	1.4442	1.6291	1.9749	1.8895	1.2010	1.3457	1.9021	2.0375
3.75	2.1916	1.6160	1.8252	2.2170	2.1200	1.3288	1.4905	2.1047	2.2501
4.00	2.4288	1.7959	2.0283	2.4643	2.3560	1.4606	1.6387	2.3170	2.4777
4.25	2.6691	1.9791	2.2418	2.7226	2.6078	1.5976	1.7885	2.5288	2.7025
4.50	2.9117	2.1717	2.4639	2.9931	2.8646	1.7361	1.9432	2.7433	2.9337
4.75	3.1562	2.3696	2.6889	3.2734	3.1329	1.8781	2.0986	2.9592	3.1670
5.00	3.4029	2.5712	2.9181	3.5562	3.4029	2.0193	2.2597	3.1818	3.4029

11.3.1 Flat and Forward Volatility (cont.)

- The **implied flat volatility** of an interest rate cap with maturity T and strike rate r_K is the level of $\sigma(r_K,T)$ in the interest rate model that exactly prices the cap

11.3.2 Forward Volatility and the Black, Derman and Toy Model

- Different implied volatilities generate different trees
  - This suggests that it is not possible to replicate, for instance, the 1-year cap by using the 2-year cap
- The full BDT is an interest rate model that is able to fit exactly all of the zero coupon bonds and all of the caps
- One difficulty with simply adding a time index i to σ in the Simple BDT model is that the tree is no longer recombining, given that:

$$z_{2,ud} = z_0 + (\theta_0 + \theta_1) \times \Delta + (\sigma_1 - \sigma_2) \times (\Delta)^{½}$$
$$z_{2,du} = z_0 + (\theta_0 + \theta_1) \times \Delta - (\sigma_1 - \sigma_2) \times (\Delta)^{½}$$

$z_{2,ud} \neq z_{2,du}$ unless $\sigma_1 = \sigma_2$

11.3.2 Forward Volatility and the Black, Derman and Toy Model (cont.)

- The full BDT solves this by using a different procedure to construct the tree:

$$z_{i,j+1} = z_{i,j} - 2 \times \sigma \times (\Delta)^{½} \qquad \text{for } j = 0,1,\ldots,i\text{-}1$$

- The implication of this equation is that instead of searching for θ_i at any step i, we search for $z_{i+1,0}$ (the top element in the interest rate tree)
- This tree is made to be recombining, so we can now do:

$$z_{i,j+1} = z_{i,j} - 2 \times \sigma_i \times (\Delta)^{½} \qquad \text{for } j = 0,1,\ldots,i\text{-}1$$

- The **forward volatility** σ_i is the level of volatility in step i in the Black, Derman, and Toy model that matches the cap price with maturity i+1
- It is possible to think of the flat volatility as a sort of weighted average of forward volatility:
    - If the forward volatility of a cap with maturity i+1 is higher than the forward volatility of a cap with maturity i, then the implied volatility of the former cap is also (likely) higher than of the latter

11.3.2 Forward Volatility and the Black, Derman and Toy Model (cont.)

Table 11.13 The Black, Derman and Toy model

$i=0$	$i=1$	$i=2$	$i=3$
		$\sigma_2 = ?$ $r_{2,uu} = ?$	...
	$\sigma_1 = ?$ $r_{1,u} = ?$		...
$r_0 = 2.17\%$		$r_{2,ud} = r_{2,uu} \times e^{-2\times\sigma_2\times\sqrt{\Delta}}$	...
	$r_{1,d} = r_{1,u} \times e^{-2\times\sigma_1\times\sqrt{\Delta}}$		...
		$r_{2,dd} = r_{2,uu} \times e^{-4\times\sigma_2\times\sqrt{\Delta}}$	...

11.3.2 Forward Volatility and the Black, Derman and Toy Model (cont.)

Table 11.14 The Black, Derman and Toy Model on November 1, 2004

Time $\Longrightarrow$	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5
Period i $\Longrightarrow$	0	1	2	3	4	5	6	7	8	9	10
$\sigma_i(\%)$ $\Longrightarrow$		18.77	18.66	27.77	30.19	29.76	30.75	33.98	31.77	31.99	30.28
node j											
0	2.17	2.68	3.21	4.26	5.37	6.45	7.97	10.58	12.02	14.63	16.36
1		2.22	2.66	3.23	3.97	4.79	5.86	7.53	8.74	10.62	12.08
2			2.21	2.44	2.94	3.56	4.31	5.36	6.36	7.72	8.93
3				1.85	2.17	2.64	3.17	3.82	4.63	5.60	6.59
4					1.60	1.96	2.33	2.72	3.37	4.07	4.87
5						1.46	1.71	1.94	2.45	2.95	3.60
6							1.26	1.38	1.79	2.15	2.66
7								0.98	1.30	1.56	1.96
8									0.95	1.13	1.45
9										0.82	1.07
10											0.79

11.4 RISK NEUTRAL TREES FOR FUTURES PRICES

- **11.4.1 Eurodollar Futures**

- **11.4.2 T-Note and T-Bond Futures**

11.4 RISK NEUTRAL TREES FOR FUTURES PRICES

- Because futures markets, such as the Eurodollar futures or the T-bond futures, are very liquid, traders extract as much information as possible from the behavior of futures prices to build risk neutral trees
- Let Fi,j(k) denote the futures price of a contract maturing at time k at time/node (i,j)
- Recall that futures are marked-to-market daily
- Assume that mark-to-market occurs at the same frequency of the time steps on the tree
- Then, the profits per period are given by the change in the futures price between one period to the next
- That is, if the interest rate moves up on the tree from ri,j to ri+1,j then the profit from the futures is N ×(Fi+1,j(k) − Fi,j(k)), where N is the contract size
- Since, by construction, the interest rate tree is risk neutral, the risk neutral expected profit from a position in futures is:

$$E^*[F_{i+1}(k) - F_i(k)] = ½ \times (F_{i+1,j}(k) - F_{i,j}) + ½ \times (F_{i+1,j+1}(k) - F_{i,j})$$

11.4 RISK NEUTRAL TREES FOR FUTURES PRICES (cont.)

- The key question is: If all market participants were risk neutral, what should the expected risk neutral profit be?
- Because it costs nothing to enter into a futures position, the answer is the expected profit should be zero
  - If the risk neutral expected profit was positive, then risk neutral agents would go infinitely long in the contract, pushing up the futures price
  - If the risk neutral expected profits from futures was negative, all risk neutral agents would short the futures
- The key implication of the risk neutral pricing methodology applied to futures is then the following restriction: $E^*[F_{i+1}(k) - F_i(k)] = 0$, which leads to:

$$F_{i,j} = ½ \times F_{i+1,j}(k) + ½ \times F_{i+1,j+1}(k)$$

- This equation allows us to move backward on the tree, exactly as we did for other securities: given the futures prices at nodes at time i+1, we can compute the futures price at node i
- Finally, at maturity, the futures price must converge to the value of the security underlying the futures contract (convergence), so:

$$F_{k,j}(\text{k}) = N \times V_{k,j},$$

- where $V_{k,j}$ is the payoff of the futures contract with maturity k (underlying security)

11.4.1 Eurodollar Futures

- For Eurodollar futures the underlying final cash payment depends on $N \times$(3-month LIBOR) where N is the contract size
- The Eurodollar futures contract with maturity k in node (i,j) is quoted as $F_{i,j}(k) = (100 - f_{i,j}(k))$ where $f_{i,j}(k)$ is the futures rate, in percentage
- The Eurodollar futures LIBOR rate at maturity (k) must converge to: $f_{k,j}(k) = N \times r_4(k,j)$, where $r_4(k,j)$ denotes a quarterly (n = 4)compounding rate (i.e. LIBOR)
- Recall that the BDT model, fitted to swaps and cap prices, generates an interest rate tree in which rates are continuously compounded, so, assuming $N = 1$, at maturity k we have:
  $f_{k,j}(k) = r_4(k,j) = 4 \times (e^{r_{k,j} \times 0.25} - 1)$
- This methodology can also be reversed, so we start with futures rates or prices (higher liquidity than swaps) in order to compute the short rate process

11.4.1 Eurodollar Futures (cont.)

Table 11.15 Eurodollar Futures Trees

3-Month Futures Rates Tree

j \ i	0	1	2	3	4
0	2.46	2.69			
1		2.23			

3-Month Eurodollar Futures Price

j \ i	0	1	2	3	4
0	97.54	97.31			
1		97.77			

6-Month Futures Rates Tree

j					
0	2.69	2.95	3.22		
1		2.44	2.67		
2			2.21		

6-Month Eurodollar Futures Price

j					
0	97.31	97.05	96.78		
1		97.56	97.33		
2			97.79		

9-Month Futures Rates Tree

j					
0	2.90	3.30	3.76	4.28	
1		2.50	2.85	3.24	
2			2.15	2.45	
3				1.86	

9-Month Eurodollar Futures Price

j					
0	97.10	96.70	96.24	95.72	
1		97.50	97.15	96.76	
2			97.85	97.55	
3				98.14	

1-Year Futures Rates Tree

j					
0	2.18	2.69	3.22	4.28	5.41
1		2.23	2.67	3.24	3.99
2			2.21	2.45	2.95
3				1.86	2.18
4					1.61

1-Year Eurodollar Futures Price

j					
0	97.82	97.31	96.78	95.72	94.59
1		97.77	97.33	96.76	96.01
2			97.79	97.55	97.05
3				98.14	97.82
4					98.39

11.4.2 T-Note and T-Bond Futures

- The party who is taking a short position in these futures, and thus commits to deliver the underlying security at maturity, is implicitly acquiring some options:
  - Quality option:
    - There are several securities that are eligible for delivery (e.g. for the 10-year contract, these are all the Treasury notes that have a maturity comprised between 6 ½ and 10 years)
    - Across all the securities that are eligible for delivery, the short trader will choose the one that is least expensive, which is then called *cheapest-to-deliver*
  - Wild card option:
    - There is a whole month to deliver the note or bond; during this month the futures contract trades until the seventh business day before the last business day
    - Every trading day in the delivery month, the short may deliver until 8 pm (Chicago time), while the contract stops trading at 2 pm (Chicago time)
    - Essentially, the trader short the contract has about 15 sequential six-hours put options during each day of the delivery month until the last day of futures trading
  - End-of-month option:
    - Trading stops seven days prior the last business day of the contract month, but delivery can occur up to the last trading day
    - Before the invoice price of the futures has been fixed on the last trading day, but bond prices keep trading, the short has a timing option as it can select any day during the last week to deliver

11.4.2 T-Note and T-Bond Futures (cont.)

- The existence of these options has an impact on the futures price itself
- For the quality option, the short trader will deliver the bond that is least expensive
  - If no correction was made to the futures price, the cheapest bond or note would correspond to the one with the lowest maturity and coupon
  - To avoid liquidity issues and manipulations, it is desirable to standardize all of the securities eligible for delivery
  - The invoice price paid by the long side to the short side equals the futures price F_{t*} at maturity t^* multiplied by a conversion factor C
    - This conversion factor is defined to make the deliverable bond comparable to a 6% coupon bond or note
- How does the short trader determine the best bond to deliver on a futures contract with price $F_{i,j}(k)$?
  - Let there be n notes that are eligible for delivery, where for each note h, $h = 1,\ldots,n$, let C^h denote its conversion factor and $P^h{}_{i,j}$ its clean price
  - For each note h we can compute the difference in price:

$$\text{Basis of note } h = P^h{}_{k,j} - F_{k,j}(k) \times C^h$$

  - The bond h with the smallest basis is the cheapest-to-deliver

11.4.2 T-Note and T-Bond Futures (cont.)

Table 11.16 Conversion Factors for 10-Year Treasury Note Futures

		Coupon	Issue Date	Maturity Date	Cusip Number	Issuance (Billions)	6% Conversion Factors Mar. 2008	Jun. 2008	Sep. 2008	Dec. 2008	Mar. 2009	Jun. 2009
1.)	@	3 1/2	02/15/08	02/15/18	912828HR4	$23.0	0.8174	0.8210	0.8244	0.8281	0.8317	0.8354
2.)		4	02/15/05	02/15/15	912828DM9	$23.0	0.8902	0.8937	—	—	—	—
3.)		4 1/8	05/16/05	05/15/15	912828DV9	$22.0	0.8941	0.8971	0.9003	—	—	—
4.)		4 1/4	11/15/04	11/15/14	912828DC1	$23.0	0.9069	—	—	—	—	—
5.)		4 1/4	08/15/05	08/15/15	912828EE6	$21.0	0.8983	0.9012	0.9040	0.9069	—	—
6.)		4 1/4	11/15/07	11/15/17	912828HH6	$21.0	0.8747	0.8771	0.8797	0.8821	0.8848	0.8873
7.)		4 1/2	11/15/05	11/15/15	912828EN6	$21.0	0.9105	0.9128	0.9153	0.9177	0.9202	—
8.)		4 1/2	02/15/06	02/15/16	912828EW6	$21.0	0.9080	0.9105	0.9128	0.9153	0.9177	0.9202
9.)		4 1/2	05/15/07	05/15/17	912828GS3	$21.0	0.8968	0.8990	0.9013	0.9034	0.9058	0.9080
10.)		4 5/8	11/15/06	11/15/16	912828FY1	$21.0	0.9095	0.9115	0.9136	0.9157	0.9179	0.9200
11.)		4 5/8	02/15/07	02/15/17	912828GH7	$21.0	0.9074	0.9095	0.9115	0.9136	0.9157	0.9179
12.)		4 3/4	08/15/07	08/15/17	912828HA1	$21.0	0.9122	0.9140	0.9158	0.9177	0.9195	0.9215
13.)		4 7/8	08/15/06	08/15/16	912828FQ8	$21.0	0.9275	0.9293	0.9310	0.9328	0.9346	0.9365
14.)		5 1/8	05/15/06	05/15/16	912828FF2	$21.0	0.9450	0.9463	0.9478	0.9491	0.9506	0.9519

Footnotes: "@" indicates the most recently auctioned U.S. Treasury security eligible for delivery.
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Source: CBOT web site: http://www.cbot.com/cbot/pub/cont detail/0,3206,1391+20356,00.html accessed on June 11, 2008.

11.4.2 T-Note and T-Bond Futures (cont.)

- An example:
  - Consider fitting the Ho-Lee model to the term structure of interest rates on January 8, 2002; resulting in an interest tree extended to longer horizons to price longer-term notes and bonds, see Table 11.17:
    - Panel A contains the zero coupon bond data on January 8, 2002 up to maturity $T = 8$
    - Panel B contains the fitted Ho-Lee model
  - Consider a 10-year Treasury note futures, with maturity of one year (December, 2003)
    - According to the terms of the contract only Treasury notes with maturities between 6 1/2 and 10 years can be delivered
  - The futures price will be adjusted by the appropriate conversion factor to make each possible deliverable bond comparable to a 6% bond
  - To illustrate the tree methodology to T-note futures, it is convenient to first consider the case in which the security underlying the future is exactly a 6%, 7-year Treasury note
  - Given the risk neutral interest rate tree in Panel B, we can compute the risk neutral tree for the deliverable T-note, which is contained in Panel C of Table 11.17
  - The T-note priced on this tree is a 8-year, 6% note, rather than 7-year note, the reason being that the T-note must have 7 years to maturity at the maturity of the futures contract, in one year from the present

11.4.2 T-Note and T-Bond Futures (cont.)

- An example (cont'd):
  - We know that at maturity the futures price must converge to the value of the underlying security to avoid an arbitrage
  - In other words, denoting by $k = 2$ the node corresponding to the maturity of the futures contract, we must have $F_{k,j}(k) = P_{k,j}$; where $P_{k,j}$ is the price of the bond at time k and node j
  - Table 11.18 shows the risk neutral futures price tree
  - Consider now the case in which in addition to the 6% note described earlier, two additional notes are available, one with a 3% coupon and one with a 9% coupon
  - Assume all of these notes have the same maturity
  - Using the same interest rate tree as in Panel B of Table 11.17 we can obtain the trees of these T-notes as well
  - They are contained in Panel A and Panel B of Table 11.19
  - The next step is to compute the conversion factor for each of these notes

$$\text{Conversion factor 3\% note} = C^1 = 0.830558903$$
$$\text{Conversion factor 6\% note} = C^2 = 1.000000000$$
$$\text{Conversion factor 9\% note} = C^3 = 1.169441097$$

11.4.2 T-Note and T-Bond Futures (cont.)

- An example (cont'd):
  - Consider now the futures maturity date k; for each node j, we can compute the basis for each bond
  - We know that at each of these nodes (k,j), the trader who is short the futures will choose to deliver the bond with the smallest basis; so we compute:
  - The futures price at time k in node j, $F_{k,j}(k)$, will move to prevent arbitrage, so that for every j the following must occur

$$\text{Node}(k, j): \min_h \left(P_{k,j}^h - F_{k,j}(k) \times C^k\right)$$

    minimization is taken across the bonds h = 1,...,n that are eligible for delivery
  - In other words, the futures price moves to make the bond price with the smallest basis in fact equal to the futures price (corrected by the conversion factor)
  - That is, $F_{k,j}(k)$ is given by:

$$\text{Node}(k, j): \min_h \left(P_{k,j}^h - F_{k,j}(k) \times C^h\right) = 0$$

  - Once $F_{k,j}$ has been computed, the rest of the risk neutral futures tree follows

$$F_{k,j}(k) = \min_h \frac{P_{k,j}^h}{C^h}$$

11.4.2 T-Note and T-Bond Futures (cont.)

- An example (cont'd):
  - Table 11.20 illustrates the calculations.
  - The last three columns report the converted bond price, namely, $P^h_{k,j} / C^h$ for each bond h (3%, 6% and 9%) and for each interest rate node j = 0, 1, 2
  - The futures price at each node j will equal the minimum across each row
  - That is, the futures price in node $(k,j) = (2,0)$ is given by $F_{2,0}(2) = 87.86$ which corresponds to the converted note price of the 3% note
  - This is the minimum across the three bonds for that particular interest rate, and thus the cheapest-to-deliver is the 3% T-note
  - Similarly, $F_{2,2}(2) = 117.28$ corresponds to the converted price of the 9% Treasury note, which in this case is the minimum across all three available notes
  - The cheapest-to-deliver is the 9% T-note
  - This finding implies that depending on whether interest rates increase or decrease, the T-note that is the cheapest-to-deliver alternates between T-notes with different coupons
  - The futures prices in the tree in Table 11.20 are always lower than the corresponding futures prices for the case only the 6% Treasury note was available
  - This lower futures price reflects the option that is implicit in the futures contract
  - The other two options (wild card option and end-of-the-month option) would also decrease the futures price.

Table 11.17 The 6% Bond Tree

Panel A. Zero Coupon Bond Data

Time T	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8
Period i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Data		99.1338	97.8925	96.1462	94.1011	91.7136	89.2258	86.8142	84.5016	82.1848	79.7718	77.4339	75.292	72.961	70.865	68.677	66.764

Panel B. The Fitted Ho-Lee Interest Rate Tree

$\theta_i(\times 100)$	1.5675	2.1824	1.4374	1.7324	0.7873	0.0423	-0.0628	0.4322	0.9271	0.1202	-0.5194	1.5300	-0.7335	1.0813	-1.0233	0.7313	-1.7140
node j																	
0	1.74	3.75	6.06	8.00	10.09	11.71	12.95	14.15	15.59	17.27	18.56	19.52	21.51	22.36	24.13	24.84	26.43
1		1.30	3.61	5.56	7.65	9.26	10.51	11.70	13.14	14.83	16.11	17.07	19.06	19.92	21.68	22.39	23.98
2			1.17	3.11	5.20	6.82	8.06	9.25	10.69	12.38	13.66	14.63	16.61	17.47	19.24	19.95	21.54
3				0.66	2.75	4.37	5.61	6.81	8.25	9.93	11.22	12.18	14.17	15.02	16.79	17.50	19.09
4					0.31	1.92	3.17	4.36	5.80	7.49	8.77	9.73	11.72	12.58	14.34	15.05	16.64
5						-0.52	0.72	1.91	3.35	5.04	6.32	7.29	9.27	10.13	11.90	12.61	14.20
6							-1.73	-0.53	0.91	2.59	3.88	4.84	6.83	7.68	9.45	10.16	11.75
7								-2.98	-1.54	0.15	1.43	2.39	4.38	5.24	7.00	7.71	9.30
8									-3.99	-2.30	-1.02	-0.05	1.94	2.79	4.56	5.27	6.86
9										-4.75	-3.46	-2.50	-0.51	0.35	2.11	2.82	4.41
10											-5.91	-4.95	-2.96	-2.10	-0.34	0.37	1.96
11												-7.39	-5.40	-4.55	-2.78	-2.07	-0.48
12													-7.85	-6.99	-5.23	-4.52	-2.93
13														-9.44	-7.68	-6.97	-5.38
14															-10.12	-9.41	-7.82
15																-11.86	-10.27
16																	-12.72

Panel C. The 6%, 8-Year Treasury Note Tree

j																	
0	106.77	96.83	88.93	82.83	78.11	74.65	72.17	70.46	69.49	69.35	70.17	71.88	74.48	78.52	83.79	90.97	100
1		112.57	102.40	94.50	88.31	83.65	80.14	77.54	75.77	74.90	75.05	76.10	78.02	81.36	85.83	92.09	100
2			118.21	108.04	100.01	93.85	89.09	85.39	82.65	80.92	80.28	80.58	81.74	84.30	87.92	93.22	100
3				123.76	113.45	105.44	99.13	94.11	90.21	87.46	85.90	85.33	85.64	87.35	90.06	94.37	100
4					128.89	118.61	110.41	103.80	98.52	94.56	91.93	90.37	89.73	90.51	92.26	95.53	100
5						133.57	123.09	114.56	107.65	102.28	98.41	95.73	94.03	93.79	94.51	96.71	100
6							137.35	126.52	117.67	110.66	105.36	101.41	98.53	97.19	96.82	97.90	100
7								139.82	128.69	119.76	112.84	107.44	103.26	100.72	99.18	99.10	100
8									140.80	129.65	120.86	113.85	108.22	104.37	101.60	100.32	100
9										140.40	129.48	120.65	113.42	108.16	104.08	101.56	100
10											138.74	127.86	118.88	112.09	106.62	102.81	100
11												135.53	124.60	116.17	109.22	104.07	100
12													130.61	120.39	111.89	105.35	100
13														124.78	114.62	106.65	100
14															117.42	107.96	100
15																109.29	100
16																	100

Table 11.18 Futures Price Tree if only a 7-year, 6% Note is Available for Delivery

Period i	0	1	2
Node j			
0	102.98	95.66	88.93
1		110.30	102.40
2			118.21

Table 11.19 3% and 6% Treasury Note Price Trees

Panel A. The 3%, 8-Year Treasury Note Tree

Period i / Node j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	86.77	79.01	72.97	68.47	65.15	62.93	61.57	60.93	60.98	61.82	63.59	66.27	69.86	74.91	81.28	89.65	100
1		93.04	85.04	78.97	74.38	71.11	68.88	67.46	66.81	67.02	68.20	70.29	73.26	77.66	83.27	90.75	100
2			99.26	91.21	85.01	80.44	77.10	74.72	73.23	72.68	73.16	74.56	76.84	80.52	85.32	91.87	100
3				105.47	97.26	91.06	86.35	82.81	80.29	78.83	78.48	79.10	80.59	83.48	87.41	93.00	100
4					111.39	103.16	96.78	91.82	88.06	85.52	84.20	83.92	84.53	86.54	89.56	94.14	100
5						116.96	108.53	101.84	96.62	92.79	90.35	89.04	88.67	89.73	91.76	95.30	100
6							121.77	113.01	106.03	100.71	96.97	94.47	93.01	93.03	94.02	96.47	100
7								125.45	116.39	109.31	104.07	100.25	97.57	96.45	96.33	97.66	100
8									127.80	118.68	111.71	106.39	102.35	100.00	98.70	98.86	100
9										128.87	119.93	112.91	107.37	103.69	101.12	100.08	100
10											128.76	119.83	112.64	107.51	103.61	101.31	100
11												127.19	118.17	111.47	106.16	102.56	100
12													123.98	115.58	108.77	103.82	100
13														119.84	111.44	105.10	100
14															114.18	106.39	100
15																107.70	100
16																	100

Panel B. The 9%, 8-Year Treasury Note Tree

Node j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	126.77	114.66	104.88	97.19	91.07	86.38	82.77	80.00	78.01	76.88	76.75	77.50	79.11	82.12	86.30	92.29	100
1		132.10	119.77	110.03	102.24	96.18	91.41	87.62	84.72	82.78	81.89	81.92	82.79	85.05	88.39	93.43	100
2			137.16	124.88	115.02	107.27	101.07	96.06	92.07	89.16	87.40	86.60	86.64	88.08	90.52	94.58	100
3				142.05	129.64	119.83	111.91	105.41	100.14	96.09	93.31	91.56	90.69	91.22	92.72	95.74	100
4					146.39	134.05	124.04	115.78	108.98	103.61	99.66	96.83	94.94	94.48	94.96	96.92	100
5						150.18	137.65	127.27	118.68	111.76	106.46	102.42	99.39	97.85	97.26	98.12	100
6							152.92	140.03	129.32	120.61	113.76	108.35	104.05	101.35	99.62	99.32	100
7								154.18	141.00	130.20	121.60	114.63	108.95	104.98	102.03	100.55	100
8									153.81	140.62	130.00	121.31	114.08	108.74	104.50	101.78	100
9										151.93	139.03	128.38	119.47	112.64	107.04	103.04	100
10											148.71	135.89	125.11	116.68	109.63	104.30	100
11												143.86	131.04	120.87	112.29	105.59	100
12													137.25	125.21	115.01	106.89	100
13														129.72	117.81	108.20	100
14															120.67	109.54	100
15																110.88	100
16																	100

Table 11.20 Treasury Note Futures when 3%, 6% and 8% Notes Are Available

Period i	0	1	2	Converted Note Prices $P^k_{i^*,j}/C^k$		
Node j				3% Note	6% Note	9% Note
0	102.48	95.12	87.86	87.86	88.93	89.68
1		109.83	102.38	102.38	102.40	102.42
2			117.28	119.51	118.21	117.28

11.5 IMPLIED TREES: FINAL REMARKS

- What do we do with this model, now?
- By construction, the model exactly prices zeros and caps, so we cannot use this model to price those

1. The model is useful for computing hedge ratios:

$$Hedge\,ratio = \frac{c_{1,u} - c_{1,d}}{V_{1,u} - V_{1,d}}$$

   where $c_{1,u}$ and $c_{1,d}$ are the values of the securities sold in the two interest rate scenarios and $V_{1,u}$ and $V_{1,d}$ are the values of the interest rate chosen to hedge the exposure
2. Once fitted to zeros and caps, the model can be used to obtain the price of other interest rate securities, such as structured notes, swaptions, American swaptions, and so on