4.5 Limit superior and limit inferior, part 1

**Definition 1.** Suppose $x_n$ is a sequence of real numbers. A point $x \in \mathbb{R}$ is called a **cluster point** of $x_n$ if for all $\epsilon > 0$, there are infinitely many $n$ such that $|x_n - x| < \epsilon$.

**Examples**

(a) For $x_n = (-1)^n$, it is easy to see that $x = \pm 1$ are the cluster points of $x_n$.

(b) The sequence $x_n = 1 + (-1)^{n+1}$ has a unique cluster point $x = 1$.

(c) The sequence $x_n = n$ has no cluster point.

(d) There is a sequence $x_n$ with three cluster points. For example, the sequence

$$x_n = \begin{cases} 
0 - \frac{1}{n}, & \text{if } n = 3k; \\
1 - \frac{1}{n}, & \text{if } n = 3k + 1; \\
2 + \frac{1}{n}, & \text{if } n = 3k + 2
\end{cases}$$

has cluster points 0, 1, 2.

(e) The sequence $x_n = \sin(n)$ has uncountably many cluster points.

(f) It is known that for a closed set $C \subset \mathbb{R}$ there exists a sequence $x_n$ such that the set of cluster points of $x_n$ is $C$. Do you think there is a sequence whose cluster points constitute $\mathbb{Q}$?

**Remark.** If $x$ is an accumulation point of the set $\{x_1, x_2, \ldots\}$, then $x$ must be a cluster point of the sequence $x_n$. However, the converse needs not be true.

**Proposition 2.** For a given sequence $x_n$, TFAE.

1. $x \in \mathbb{R}$ is a cluster point of $x_n$.

2. $\forall \epsilon > 0$ and $\forall K \in \mathbb{N}$, $\exists n \geq K$ such that $|x_n - x| < \epsilon$.

3. There exists a subsequence $x_{n_k}$ of $x_n$ such that $x_{n_k} \to x$. 

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Proof. (1) → (2): Suppose $\epsilon > 0$ and $K \in \mathbb{N}$ are given. Then since $x$ is a cluster point of $x_n$, there are infinitely many $n$ such that $|x_n - x| < \epsilon$. Thus, at least one such $n$ should be greater than $K$.

(2) → (3): We use (2) with $\epsilon = 1$ and $K = 1$, and get $n_1$ such that $|x_{n_1} - x| < 1$. Then use (2) with $\epsilon = 1/2$ and $K = n_1 + 1$, and get $n_2 > n_1$ such that $|x_{n_2} - x| < 1/2$.

Suppose that we have found $n_1 < n_2 < \cdots < n_k$ such that $|x_{n_j} - x| < 1/j$ for all $j = 1, 2, \ldots, k$. Then from (2) with $\epsilon = 1/(k + 1)$ and $K = n_k + 1$, we see that there exists $n_{k+1} > n_k$ such that $|x_{n_{k+1}} - x| < 1/(k + 1)$. By repeating this process, one can get a subsequence $x_{n_k}$ such that $|x_{n_k} - x| < 1/k$. Then definitely $x_{n_k} \to x$, as we wished.

(3) → (1): For any given $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that $k \geq K$ implies $|x_{n_k} - x| < \epsilon$ (\because $x_{n_k} \to x$). Therefore infinitely many $x_n$'s ($= x_{n_K}, x_{n_{K+1}}, x_{n_{K+2}}, \ldots$) are in $N(x, \epsilon)$. \hfill $\square$

Theorem 3. A sequence $x_n$ converges to $x$ if and only if $x_n$ is bounded and $x$ is the only cluster point of $x_n$.

Proof. ($\longrightarrow$) Since $x_n$ converges to $x$, $x_n$ is definitely bounded and $x$ is a cluster point of $x_n$. Thus all we need to show is that $x$ is the unique cluster point of $x_n$. But if $y \in \mathbb{R}$ is a cluster point of $x_n$, there exists a subsequence $x_{n_k}$ that converges to $y$. But $x_{n_k}$ also converges to $x$ since $x_n$ converges to $x$, hence we have $x = y$ by the uniqueness of the limit.

($\longleftarrow$) Suppose $x_n \not\to x$. Then $\exists \epsilon > 0$ and a subsequence $x_{n_k}$ such that $|x_{n_k} - x| \geq \epsilon$ for all $k = 1, 2, \ldots$. But since $x_{n_k}$ is bounded, there is a sub-subsequence $x_{n_{k_j}}$ (i.e., a subsequence of $x_{n_k}$) such that $x_{n_{k_j}} \to y$ for some $y \in \mathbb{R}$. This means that $y$ is a cluster point of $x_n$, hence $x = y$ by our assumption. But this gives a contradiction, since

$$\epsilon \leq |x_{n_{k_j}} - x| < \frac{\epsilon}{2},$$

for sufficiently large $j$. This completes the proof. \hfill $\square$

Definitions for limsup and liminf

Suppose $x_n$ is a bounded sequence in $\mathbb{R}$. Let $A_k = \{x_k, x_{k+1}, x_{k+2}, \ldots\}$ and $s_k = \sup(A_k)$. Since $x_n$ is bounded, the sequence $s_k$ is well-defined and
bounded below. Moreover, \( s_k \) is decreasing because \( A_1 \supset A_2 \supset A_3 \supset \cdots \). In other words, \( s_k \) is decreasing and bounded below, hence convergent. Now we define the **limit superior** (상극한) of \( x_n \) by

\[
\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{k \to \infty} s_k = \inf \{ s_k : k \in \mathbb{N} \} = \inf \sup_{n \geq k} \{ x_n \}.
\]

Similarly, if we define \( l_k = \inf(A_k) \), then \( l_k \) is increasing and bounded above, hence convergent. Thus we define the **limit inferior** (하극한) of \( x_n \) by

\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{k \to \infty} l_k = \sup \{ l_k : k \in \mathbb{N} \} = \sup \inf_{n \geq k} \{ x_n \}.
\]

If \( x_n \) is not bounded above, we define \( \limsup x_n = \infty \). Note that in this case, \( s_k = \infty \) for all \( k \). If \( x_n \to -\infty \), then we define \( \limsup x_n = -\infty \). In this case the sequence \( s_k \) is well-defined, but it is not bounded below. If \( x_n \) is bounded above, not bounded below, and not going to \( -\infty \), then one can check that \( s_k \) is a convergent sequence and we define \( \limsup x_n = \lim s_k \) as before.

Similarly we define \( \liminf x_n = -\infty \) if it is not bounded below, and \( \liminf x_n = \infty \) if \( x_n \to \infty \). Otherwise we define \( \liminf x_n = \lim l_k \).

**Remark**

(a) Note that the expressions \( \limsup x_n = \inf_k \sup_{n \geq k} \{ x_n \} \) and \( \liminf x_n = \sup_k \inf_{n \geq k} \{ x_n \} \) are still valid even when \( x_n \) is unbounded, whereas the sequences \( s_k \) and \( l_k \) may not be well-defined in this case.

(b) For any sequence \( x_n \), the limit superior and limit inferior always exist, while the limit itself may not.