Chapter 10 Vector Integral Calculus

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Chapter 10 Vector Integral Calculus

10.1 Line Integrals

A definite integral

$$\int_a^b f(x)dx$$

$\rightarrow$ Integrate the *integrand* $f(x)$ from $x=a$ to $x=b$

A *line integral* (or *curve integral*)

$\rightarrow$ integration along a curve *C*.

In parametric representation,

$$\boxed{\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}}. \qquad : \textit{path of integration}$$

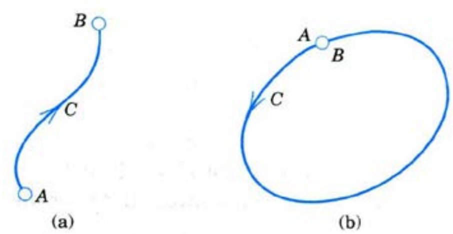


Fig. 217. Oriented curve

In the *positive direction* of *C*, the parameter *t* increases.
C is a *smooth curve* if $\vec{r}'(t)$ is continuous.
In a *closed curve* the initial and the terminal points coincide.
A *piecewise smooth* curve has finitely many smooth curves.

Definition and Evaluation of Line Integrals

A *line integral* of a vector function $\vec{F}(\vec{r})$ over a curve *C*

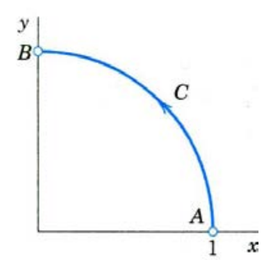
$$\boxed{\int_c \vec{F}(\vec{r})\bullet d\vec{r}=\int_a^b \vec{F}(\vec{r}(t))\bullet\frac{d\vec{r}(t)}{dt}dt}$$

Since $d\vec{r}=dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}$

$$\boxed{\int_c \vec{F}(\vec{r})\bullet d\vec{r}=\int_c (F_1dx+F_2dy+F_3dz)=\int_a^b\left(F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt}\right)dt}$$

<u>Example 1</u> Line integral in the plane

Find the line integral of $\vec{F}(\vec{r})=-y\hat{i}-xy\hat{j}$ over the circular arc.



solution:

$\vec{r}(t)=\cos t\ \hat{i}+\sin t\ \hat{j}$, $\qquad : \ 0\le t\le\frac{\pi}{2}$

$\vec{r}'(t)=-\sin t\ \hat{i}+\cos t\ \hat{j}$

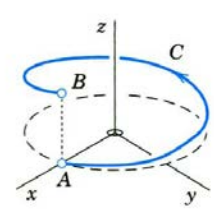
$\rightarrow\ \vec{F}(\vec{r})=-\sin t\ \hat{i}-\cos t\sin t\ \hat{j}$

$$\rightarrow\ \int_c \vec{F}\bullet d\vec{r}=\int_0^{\frac{\pi}{2}}\left(-\sin t\ \hat{i}-\cos t\sin t\ \hat{j}\right)\bullet\left(-\sin t\ \hat{i}+\cos t\ \hat{j}\right)dt\ \Rightarrow\ \int_0^{\frac{\pi}{2}}\left(\sin^2 t-\cos^2 t\sin t\right)dt\Rightarrow\frac{\pi}{4}-\frac{1}{3}$$

<u>Example 2</u> Line integral in space

Find the line integral of $\vec{F}$ along a helix *C*.

$\vec{F}=z\hat{i}+x\hat{j}+y\hat{k}$, $\qquad C:\ \vec{r}(t)=\cos t\ \hat{i}+\sin t\ \hat{j}+3t\ \hat{k}$, $\quad (0\le t\le 2\pi)$



$$\rightarrow\ \vec{F}\bullet\vec{r}'(t)=\left[3t\ \hat{i}+\cos t\ \hat{j}+\sin t\ \hat{k}\right]\bullet\left[-\sin t\ \hat{i}+\cos t\ \hat{j}+3\hat{k}\right]$$

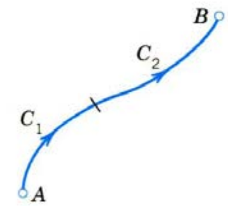
$$\int_c \vec{F}\bullet d\vec{r}=\int_0^{2\pi}\left(-3t\sin t+\cos^2 t+3\sin t\right)dt\Rightarrow 7\pi$$

Simple general properties of the line integrals

$$\int_C k\vec{F} \bullet d\vec{r} = k\int_C \vec{F} \bullet d\vec{r} \qquad : k, \text{ constant}$$

$$\int_C \left(\vec{F} + \vec{G}\right) \bullet d\vec{r} = \int_C \vec{F} \bullet d\vec{r} + \int_C \vec{G} \bullet d\vec{r}$$

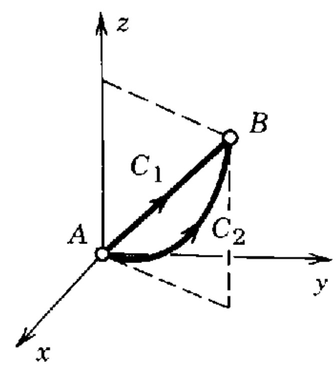
$$\int_C \vec{F} \bullet d\vec{r} = \int_{C_1} \vec{F} \bullet d\vec{r} + \int_{C_2} \vec{F} \bullet d\vec{r}$$



Path Dependence

Theorem 2 Path Dependence

The line integral generally depends not only on $\vec{F}$ and end points of the path, but also on the path itself.



Example Dependence of a line integral on path

$$\vec{F} = 5z\hat{i} + xy\hat{j} + x^2 z\hat{k}$$

Two curves with same end points $A : (0,0,0)$, $B : (1,1,1)$,

C_1 : straight line segment : $\vec{r}_1(t) = t\hat{i} + t\hat{j} + t\hat{k}, \quad 0 \le t \le 1$

C_2 : parabolic arc : $\vec{r}_2(t) = t\hat{i} + t\hat{j} + t^2\hat{k}, \quad 0 \le t \le 1$

solution:

$$\vec{F}\left(\vec{r}_1(t)\right) = 5t\hat{i} + t^2\hat{j} + t^3\hat{k}, \qquad \vec{r}_1' = \hat{i} + \hat{j} + \hat{k}$$

$$\rightarrow \int_{C_1} \vec{F} \bullet d\vec{r} = \int_0^1 (5t + t^2 + t^3)dt = \frac{1}{4}$$

Similarly

$$\vec{F}(\vec{r}_2(t)) = 5t^2\hat{i} + t^2\hat{j} + t^4\hat{k}, \qquad \vec{r}_2' = \hat{i} + \hat{j} + 2t\hat{k}$$

$$\rightarrow \int_{C_2} \vec{F} \bullet d\vec{r} = \int_0^1 (5t^2 + t^2 + 2t^5)dt = \frac{2}{3}$$

In general, a line integral depends on $\vec{F}$, A, B and C

10.2 Path Independent of Line Integrals

A line integral

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) \qquad d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

We shall find the condition under which the integral is *path independent*.

Theorem 1 Independence of Path

> A line integral $\int_C \vec{F} \bullet d\vec{r}$ is <u>path independent</u> in a domain *D*
> if and only if $\vec{F}$ is the <u>gradient of a scalar function</u> *f* in *D*.
> ↑ $\vec{F} = \nabla f$

Proof :

Let $\vec{F} = \nabla f$ and $C:\ \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $\qquad a \le t \le b$

$$\rightarrow \int_C \vec{F} \bullet d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz \Rightarrow \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dz + \frac{\partial f}{\partial z} dz \right) \Rightarrow \int_a^b \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \right) dt \Rightarrow \int_a^b \frac{\partial f}{\partial t} dt$$

$$\Rightarrow f\left[x(t),\ y(t),\ z(t)\right]\Big|_{t=a}^{t=b} \Rightarrow f\left(x(b),\ y(b),\ z(b)\right) - f\left(x(a),\ y(a),\ z(a)\right) \Rightarrow f(B) - f(A)$$

The below formula can be just used if the path independence is already known,

$$\boxed{\int_A^B \left(F_1 dx + F_2 dy + F_3 dz\right) = f(B) - f(A)}, \qquad \text{where } \vec{F} = \nabla f,\ f \text{ is potential}$$

<u>Example 1</u> Path Independence

Show that $\int_C \vec{F} \bullet d\vec{r} = \int_C \left(2xdx + 2ydy + 4zdz\right)$ is path independent and

find its value for end points $A:(0,0,0)$ and $B:(2,2,2)$.

$\vec{F}$ is the gradient of $f = x^2 + y^2 + 2z^2$. : The integral is path independent.

Its value is

$$f(B) - f(A) = f(2,2,2) - f(0,0,0) = 4 + 4 + 8 = 16$$

<u>Example 2</u> Path Independence

Find $I = \int_C (3x^2 dx + 2yzdy + y^2 dz)$ from *A* :(0, 1, 2) to *B* :(1, -1, 7) by showing $\vec{F}$ has a potential.

Let $\vec{F} = \nabla f \quad \rightarrow \quad F_1 = \frac{\partial f}{\partial x} = 3x^2, \quad F_2 = \frac{\partial f}{\partial y} = 2yz, \quad F_3 = \frac{\partial f}{\partial z} = y^2$

$\rightarrow \quad f = x^3 + g(y,z), \quad \frac{\partial g}{\partial y} = 2yz$ and $g = y^2 z + h(z), \quad f = x^3 + y^2 z + h(z), \quad \frac{\partial f}{dz} \Rightarrow y^2 + \frac{\partial h}{\partial z} = y^2$ and $h = const.$

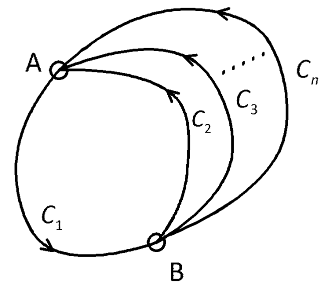
$\rightarrow \quad f = x^3 + y^2 z + C$

Then, $\quad I = f(1,-1,7) - f(0,1,2) \Rightarrow 6$

Path Independence and Integration Around Closed Curves

Theorem 2 Path Independence

> A line integral of $\vec{F}$ is path independent in D
> if and only if every closed line integral of $\vec{F}$ is zero in D



Proof :
The line integral is zero for every closed path.

$$\int_{C_1} \vec{F} \bullet d\vec{r} + \int_{C_2} \vec{F} \bullet d\vec{r} = 0\,, \quad \int_{C_1} \vec{F} \bullet d\vec{r} + \int_{C_3} \vec{F} \bullet d\vec{r} = 0\,, \quad \ldots.$$

Since $\int_{C_1} ()$ is common

$\rightarrow \int_{C_2} \vec{F} \bullet d\vec{r} = \int_{C_3} \vec{F} \bullet d\vec{r} = \ldots.$: Path independent

$\vec{F}$ is called *conservative*, in this case

10.3 Double integrals

$$\lim_{D \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k \equiv \iint_R f(x,y) dxdy$$

↑ ↑ ↑
Max. diagonal of rectangles
A point inside the k-th rectangle
Area of the k-th rectangle

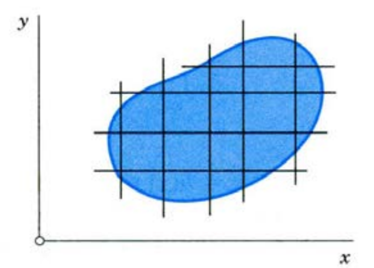


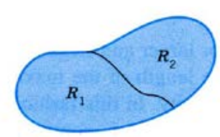
Fig. 225. Subdivision of a region R

- Properties of double integrals

$$\iint_R kf\ dxdy = k \iint_R f\ dxdy$$

$$\iint_R (f+g)\ dxdy = \iint_R f\ dxdy + \iint_R g\ dxdy$$

$$\iint_R f\ dxdy = \iint_{R_1} f\ dxdy + \iint_{R_2} f\ dxdy\,, \quad \text{where } R = R_1 + R_2$$



- Mean value theorem

$$\iint_R f(x,y)\ dxdy = f(x_0, y_0) A$$

↑ ↑ Area of R
A point in R

Evaluation of double integrals

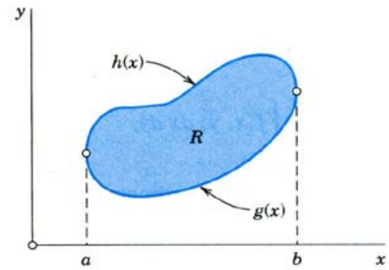
R is given as $a \le x \le b, \quad g(x) \le y \le h(x)$

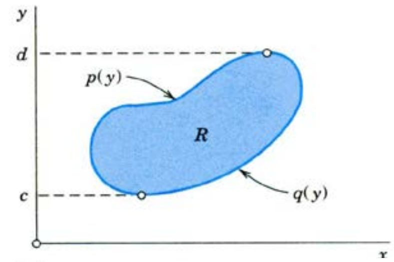
The integration of f over of R

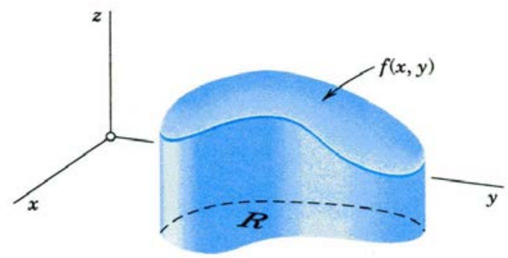
$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)dy \right] dx$$

Similarly, when R is given by $c \le y \le d, \quad p(y) \le x \le q(y)$

$$\iint_R f(x,y)dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)dx \right] dy$$







Application of double integral

Area $A = \iint_R dxdy$

Volume above R and beneath the surface $z = f(x,y)$

$$V = \iint_R f(x,y)dxdy$$

Change of Variables in Double Integral

For a definite integral

$$\int_a^b f(x)dx = \int_\alpha^\beta f\left(x(u)\right)\frac{dx}{du}du ,$$

$x(u)$ is continuous and has a continuous derivative

For a double integral

$$\iint_R f(x,\ y)dxdy = \iint_R f\left(x(u,v),\ y(u,v)\right)\left|J\right|dudv$$

The *Jacobian* $J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

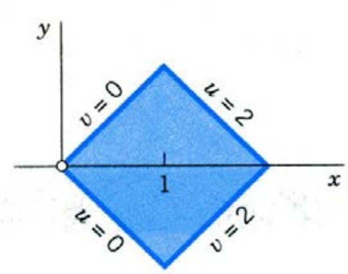
- Change to polar coordinates, $x = r\cos\theta, \quad y = r\sin\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \Rightarrow r \qquad \rightarrow \qquad \iint_R f\left(x,y\right)dxdy = \iint_{R^*} f\left(r\cos\theta,\ r\sin\theta\right)rdrd\theta$$

<u>Example 1</u> Change of Variables in a Double Integral

Evaluate the double integral over the square R shown in the right.

$$\iint_R (x^2 + y^2)dxdy$$



Change of variables

$$x + y = u, \quad x - y = v \qquad \rightarrow \qquad x = \tfrac{1}{2}(u+v), \quad y = \tfrac{1}{2}(u-v)$$

The Jacobian $J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$

R corresponds to $0 \le u \le 2, \quad 0 \le v \le 2$

$$\rightarrow \iint_R (x^2 + y^2)dxdy = \int_0^2\int_0^2 \tfrac{1}{2}\left(u^2 + v^2\right)\tfrac{1}{2}dudv = \frac{8}{3}$$

10.4 Green's Theorem in the plane

Theorem 1 Green's theorem in the plane
(Transformation between Double Integrals and Line Integrals)

R : closed bounded region in xy-plane
C : boundary of R

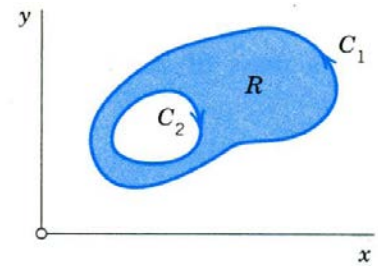
Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right)$$

In vectorial form

$$\iint_R \left(\nabla \times \vec{F} \right) \bullet \hat{k} \; dxdy = \oint_C \vec{F} \bullet d\vec{r}$$

(Right hand rule between $\hat{k}$ and C)



Proof : skip

Example 1

$F_1 = y^2 - 7y$, $\quad F_2 = 2xy + 2x \quad\quad C$: a circle $\; x^2 + y^2 = 1$

Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy \Rightarrow \iint_R \left[(2y+2) - (2y-7) \right] dxdy \Rightarrow 9 \iint_R dxdy \Rightarrow 9\pi$$

The curve is represented

$\vec{r}(t) = \cos t \; \hat{i} + \sin t \; \hat{j} \quad\quad 0 \le t \le 2\pi$

$\vec{r}'(t) = -\sin t \; \hat{i} + \cos t \; \hat{j}$

On C

$F_1 = \sin^2 t - 7\sin t$, $\quad F_2 = 2\cos t \sin t + 2\cos t$

$$\oint_C \vec{F} \bullet d\vec{r} = \oint_C \vec{F} \bullet \vec{r}\,' dt \Rightarrow \int_0^{2\pi} \left[\left(\sin^2 t - 7\sin t \right)\left(-\sin t \right) + \left(2\cos t \sin t + 2\cos t \right) \cos t \right] dt \Rightarrow 9\pi$$

10.5 Surfaces for Surface Integrals

Representations of surfaces

In xyz-space

$z = f(x,y)$ or $g(x,y,z) = 0$

- Parametric representation

$$\boxed{\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}}, \qquad : (u,v) \text{ in } R$$

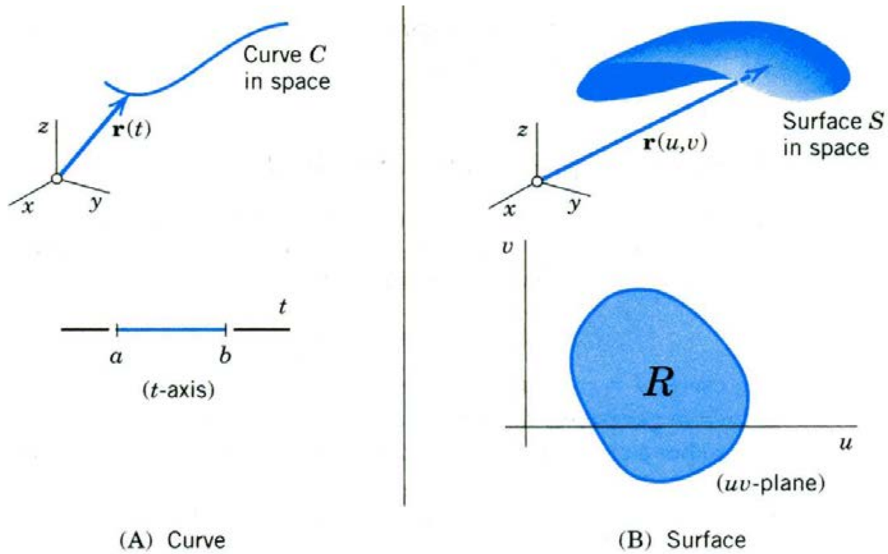


Fig. 239. Parametric representations of a curve and a surface

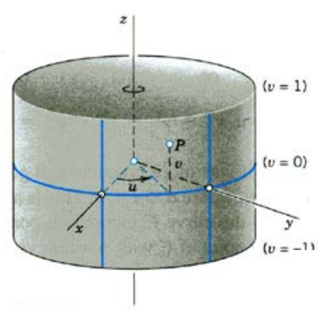
Example 1 Parametric Representation of a Cylinder

$x^2 + y^2 = a^2, \quad -1 \le z \le 1$

Parametric representation

$\vec{r}(u,v) = a\cos u\,\hat{i} + a\sin u\,\hat{j} + v\hat{k}$

where $0 \le u \le 2\pi, \quad -1 \le v \le 1$



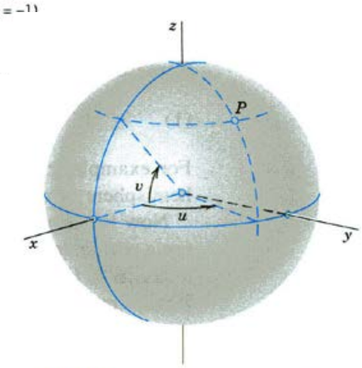
Example 2 Parametric Representation of a Sphere

$x^2 + y^2 + z^2 = a^2$

Parametric representation (different from that using spherical coordinates.)

$\vec{r}(u,v) = a\cos v\cos u\,\hat{i} + a\cos v\sin u\,\hat{j} + a\sin v\,\hat{k}$

where $0 \le u \le 2\pi$ for meridian, $-\dfrac{\pi}{2} \le v \le \dfrac{\pi}{2}$ for parallel

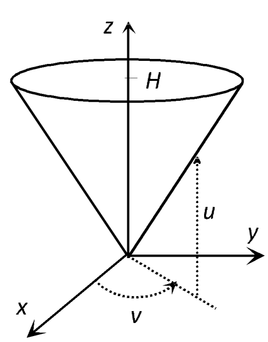


Example 3 Parametric Representation of a Cone

$z = \sqrt{x^2 + y^2}, \quad 0 \le z \le H$

$\vec{r}(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u\,\hat{k},$

where $0 \le u \le H, \; 0 \le v \le 2\pi$



Tangent Plane and Surface Normal

Tangent plane of S at P : Plane containing tangent vectors of S at P
Normal vector of S at P : A vector perpendicular to the tangent plane

- Parametric representation of S: $\vec{r}(u,v)$

  A curve C on S : $\vec{r}\left(u(t),\ v(t)\right) \equiv \vec{r}_1(t)$

  A *tangent vector* of C : $\vec{r}_1{}'(t) = \frac{d\vec{r}_1}{dt} \Rightarrow \underline{\frac{\partial \vec{r}}{\partial u}}u' + \underline{\frac{\partial \vec{r}}{\partial v}}v'$

  ↑ ↑ Tangent vectors of S at P

  A *normal vector* of S at P : $\boxed{\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \equiv \vec{r}_u \times \vec{r}_v}$

  The unit normal vector : $\hat{n} = \frac{\vec{N}}{|\vec{N}|}$

- When S is given by $g(x,y,z)=0$

  The surface normal vector : $\boxed{\vec{N} = \nabla g}$

  (Theorem 2 in 9.7)

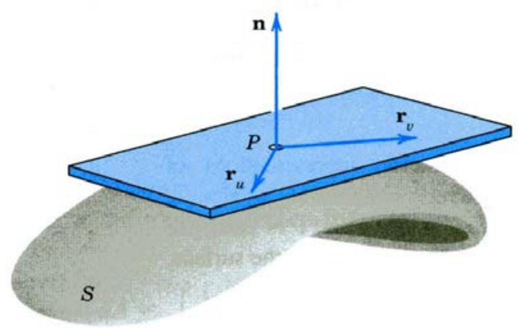


Fig. 242. Tangent plane and normal vector

Example 4 Unit Normal Vector of a Sphere

$$g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$$

$$\rightarrow \quad \text{grad}\, g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\rightarrow \quad \hat{n} = \frac{x}{a}\hat{i} + \frac{y}{a}\hat{j} + \frac{z}{a}\hat{k}$$

Example 5 Unit Normal Vector of a Cone

$$g\left(x,y,z\right) = -z + \sqrt{x^2+y^2}$$

$$\rightarrow \quad \nabla g = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j} - \hat{k}$$

$$\rightarrow \quad \hat{n} = \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j} - \hat{k}\right)$$

10.6 Surface Integrals

A surface S in parametric representation : $\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$

The surface normal vector : $\boxed{\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \equiv \vec{r}_u \times \vec{r}_v}$, $\hat{n} = \frac{1}{|\vec{N}|}\vec{N}$ (unit normal vector)

The element of area of S : $\boxed{\vec{N}dudv \Rightarrow \hat{n}|\vec{r}_u \times \vec{r}_v|dudv \Rightarrow \hat{n}|\vec{r}_u du \times \vec{r}_v dv| \Rightarrow \hat{n}dA}$

A surface integral of $\vec{F}(\vec{r})$ over S

$$\boxed{\iint_S \vec{F} \bullet d\vec{A} \equiv \iint_S \vec{F} \bullet \hat{n}dA = \iint_R \vec{F} \bullet \vec{N}\, dudv}$$

$\vec{F} \bullet \hat{n}$ is the normal component of $\vec{F}$.

If $\vec{F} = \rho\vec{v}$ (Volume density of fluid $\times$ Velocity) $\rightarrow$ The integral gives the *flux* across S (Mass of fluid crossing S per unit time)

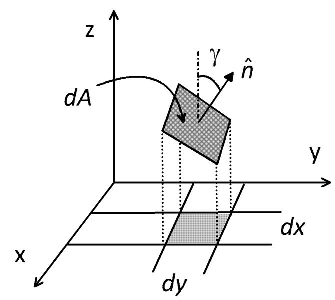
$\uparrow$

Flux density(Mass per unit time per unit area)

- The surface integral in component form

  Using $\vec{F} = [F_1, F_2, F_3]$, $\vec{N} = [N_1, N_2, N_3]$ and $\hat{n} = [\cos\alpha, \cos\beta, \cos\gamma]$

$$\begin{aligned}\iint_S \vec{F} \bullet d\vec{A} &= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)dA \\ &= \iint_S (F_1\, dydz + F_2\, dzdx + F_3\, dxdy) \\ &= \iint_R (F_1N_1 + F_2N_2 + F_3N_3)dudv\end{aligned}$$



Projection of dA onto xy-plane

$= dA\cos\gamma = dxdy$

Note

$\cos\alpha\, dA = dydz$, $\cos\beta\, dA = dzdx$, $\cos\gamma\, dA = dxdy$

α, β, γ : *angle between* $\hat{n}$ *and coordinate axes*

9.7 Triple Integrals. Divergence Theorem of Gauss

A *triple integral* of *f* over region *T*

$$\lim_{D\to 0}\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k \equiv \iiint_T f(x,y,z)dxdydz \quad \text{or} \quad \iiint_T f(x,y,z)dV$$

↑ ↑ ↑

Max. diagonal of cubes

A point in the *k*-th cube

Volume of the *k*-th cube

Divergence Theorem of Gauss

Triple integral ↔ Closed surface integral

Theorem 1 Divergence Theorem of Gauss

(Transformation between Triple and Surface Integrals)

T is a closed bounded region in space whose boundary is a piecewise smooth orientable surface *S*.

$\vec{F}(x,y,z)$ is continuous and has continuous first part. derivative in *T*.

Then

$$\iiint_T \nabla \bullet \vec{F}\, dV = \oiint_S \vec{F} \bullet d\vec{A}$$

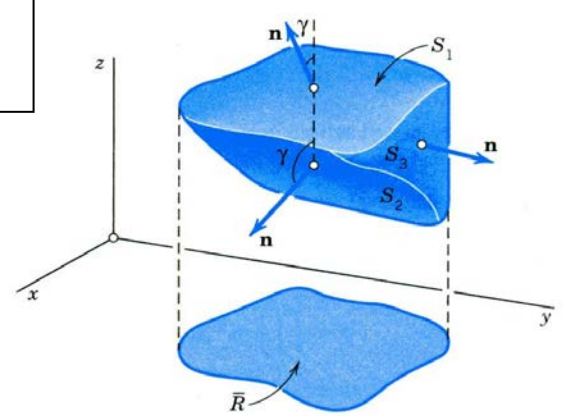
$d\vec{A} = \hat{n}dA$, $\hat{n}$ is the outer unit surface normal vector

$$\rightarrow \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \oiint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)dA$$

$$= \iint_S F_1 dydz + F_2 dzdx + F_3 dxdy$$

where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$

Prove : skip



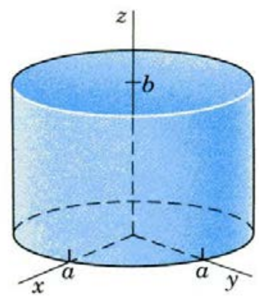
<u>Example 1</u> Evaluation of a Surface Integrals by the Divergence Theorem

Evaluate

$$I = \iint_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy)$$

on a closed surface of the cylinder, $x^2 + y^2 = a^2 \quad (0 \le z \le b)$,

with top and bottom disks, $z = 0$ and $z = b \quad (x^2 + y^2 \le a^2)$.



The surface integral

$$\iint_R F_1 dydz + F_2 dzdx + F_3 dxdy \Rightarrow \iiint_T \nabla \bullet \vec{F}\ dV \Rightarrow \iiint_T 5x^2\ dV \Rightarrow \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \left(5r^2 \cos^2\theta\right) r dr\, d\theta\, dz$$

↑ $F_1 = x^3$, $F_2 = x^2 y$, $F_3 = x^2 z$.
$\rightarrow \quad \nabla \bullet \vec{F} = 3x^2 + x^2 + x^2 \Rightarrow 5X^2$.

↑ Volume integral in cylindrical coordinates
$(x = r\cos\theta, \quad y = r\sin\theta)$

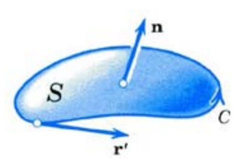
10.9 Stokes's Theorem

<u>Theorem 1</u> Stokes's Theorem
(Transformation between Surface and Line Integrals)

Let S be a piecewise smooth oriented surface.
Let its boundary be a piecewise smooth simple closed curve C.
Let $\vec{F}(x,y,z)$ be a continuous vector function with continuous first partial derivatives.

$$\boxed{\iint_S \left(\nabla \times \vec{F}\right) \bullet d\vec{A} = \oint_C \vec{F} \bullet d\vec{r}}$$

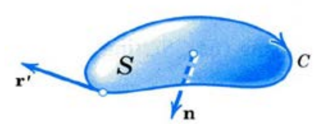
$d\vec{A} = \hat{n}\, dA, \quad d\vec{r} = \vec{r}\,'(s) ds$



Using $\hat{n}dA = \vec{N}dudv$

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right] dudv$$

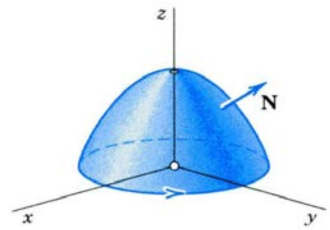
$$= \oint_C \left(F_1 dx + F_2 dy + F_3 dz\right)$$



Proof: skip!

<u>Example 1</u> Verification of Stokes's Theorem

Let $\vec{F} = [y,\ z,\ x]$ and S is the paraboloid, $z = f(x,y) = 1-(x^2+y^2)$ for $z \ge 0$.



The curve

$$\vec{r}(s) = [\cos s,\ \sin s,\ 0]$$

$$\rightarrow \quad \vec{r}\,'(s) = [-\sin s,\ \cos s,\ 0]$$

On the curve C

$$\vec{F} = [y,\ z,\ x] = [\sin s,\ 0,\ \cos s]$$

Hence

$$\oint_C \vec{F} \bullet d\vec{r} \Rightarrow \int_0^{2\pi} \vec{F}(\vec{r}(s)) \bullet \vec{r}\,'(s) ds \Rightarrow \int_0^{2\pi} [(\sin s)(-\sin s) + 0 + 0] ds = -\pi$$

- The surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} x \\ y & z & x \end{vmatrix} \Rightarrow -\hat{i} - \hat{j} - \hat{k}$$

$$\vec{N} = \nabla(z - f(x,y)) \Rightarrow 2x\,\hat{i} + 2y\,\hat{j} + \hat{k}$$

$$\iint_S (\nabla \times \vec{F}) \bullet d\vec{A} = \iint_R (\nabla \times \vec{F}) \bullet \vec{N}\, dxdy \Rightarrow \iint_R (-2x - 2y - 1)\, dxdy \Rightarrow \int_{\theta=0}^{2\pi} \int_{r=0}^{1} [-2r(\cos\theta + \sin\theta) - 1] r dr d\theta = -\pi$$

↑

In cylindrical coordinates