Chapter 9 Vector Differential Calculus, Grad, Div, Curl

9.5 Curves. Arc length. Curvature. Torsion

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9.9 Curl of a Vector Field
9.5 Curves. Arc length. Curvature. Torsion

A major application of vector calculus concerns curves and surfaces.

A curve $C$ can be represented by a vector function with a parameter $t$.

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

: parametric representation

The direction of the curve is determined by increasing values of $t$.

- Another representation of a curve
  $$x, \quad y = f(x), \quad z = g(x)$$
  projection of $C$ onto $xy$ plane
  projection of $C$ onto $xz$ plane

Another representation of $C$ can be given by an intersection of two surfaces

$$F(x, y, z) = 0, \quad G(x, y, z) = 0$$

**Example 1  Circle**

The circle $x^2 + y^2 = 4, \quad z = 0$ is given

In parametric representation

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = [2\cos t, 2\sin t, 0]$$

: $0 \leq t \leq 2\pi$

Check

$$x^2 + y^2 = (2\cos t)^2 + (2\sin t)^2 = 4$$

For $t=0$ : $\mathbf{r}(0) = [2, 0, 0]$

For $t = \pi/2$ : $\mathbf{r}(\pi/2) = [0, 2, 0]$

... As $t$ increases, $\mathbf{r}(t)$ moves counterclockwise.
Example 2  Ellipse
An ellipse in the form of vector function
\[ \mathbf{r}(t) = [x(t), y(t), z(t)] = [a \cos t, b \sin t, 0] \]
\[ \rightarrow x = a \cos t, \quad y = b \sin t \]
\[ \rightarrow (\cos t)^2 + (\sin t)^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{: Conventional form of an ellipse} \]

Example 3  straight line
A straight line in the direction of a unit vector \( \mathbf{b} \).
It passes through a point A.
\( \mathbf{a} \) is a position vector.

For a point on the line is given by a position vector
\[ \mathbf{r}(t) = \mathbf{a} + t \mathbf{b} \Rightarrow [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3] \]

- Curves
  Plane curve : a curve in a plane.
  Twisted curve : the others.
  Simple curve : a curve without multiple points at which the curve intersects or touches itself.
  Arc of a curve : a portion between two points in the curve, simply called 'curve'.

Example 3  circular helix
A circular helix in parametric representation
\[ \mathbf{r}(t) = [a \cos t, b \sin t, ct] \]
\[ c > 0 \quad \text{: right handed screw} \]
\[ c < 0 \quad \text{: left handed screw} \]
\[ c = 0 \quad \text{: circle} \]
**Tangent to a Curve**

A *tangent* is a straight line touching a curve.

A vector passing through both $P$ and $Q$ is

$$\frac{1}{\Delta t} \left[ \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \right]$$

Take the limit

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \right]$$

↑

*Tangent vector* of $C$ at $P$

The unit tangent vector

$$\hat{u} = \frac{\mathbf{r}'}{\| \mathbf{r}' \|} : \text{Direction of increasing } t$$

- The function for the tangent of $C$ at $P$

  $$\mathbf{q}(w) = \mathbf{r} + w \mathbf{r}' : \mathbf{r} \text{ and } \mathbf{r}', \text{ constant vectors. } w, \text{ parameter}$$

**Example 5  Tangent to an Ellipse**

Find the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P: (\sqrt{2}, 1/\sqrt{2})$

An ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow a=2, \ b=1$

The parametric presentation $\rightarrow \mathbf{r}(t) = [2\cos t, \sin t, 0]$

At $P: (\sqrt{2}, 1/\sqrt{2}) \rightarrow t = \pi / 4$

The derivative $\rightarrow \mathbf{r}'(t) = [-2\sin t, \cos t, 0]$

At $P \rightarrow \mathbf{r}'(\pi / 4) = [\sqrt{2}, 1/\sqrt{2}, 0]$

The tangent is

$$\mathbf{q}(w) = [\sqrt{2}, 1/\sqrt{2}] + w[-\sqrt{2}, 1/\sqrt{2}] \Rightarrow [\sqrt{2}(1-w), (1/\sqrt{2})(1+w)]$$
Length of a Curve

The length of the curve

\[ I = \lim_{N \to \infty} \sum_{n=1}^{N} \left\| \mathbf{r}(a + n\Delta t) - \mathbf{r}(a + (n-1)\Delta t) \right\| \]

\[ \Rightarrow \lim_{\Delta t \to 0} \sum_{n=1}^{N} \frac{\left\| \mathbf{r}(a + n\Delta t) - \mathbf{r}(a + (n-1)\Delta t) \right\|}{\Delta t} \]

\[ \Rightarrow \int_{a}^{b} \left\| \mathbf{r}'(t) \right\| dt \]

Hence

\[ I = \int_{a}^{b} \left\| \mathbf{r}'(t) \right\| dt = \int_{a}^{b} \sqrt{\mathbf{r}' \cdot \mathbf{r}''} dt \]

Arc Length of a Curve

The upper limit is now a variable

\[ s(t) = \int_{a}^{t} \sqrt{\mathbf{r}' \cdot \mathbf{r}''} dt \]

: Arc length of \( C \).

\[ \mathbf{r}' = \frac{d\mathbf{r}}{dt} \]

(11)

Linear element \( ds \)

Differentiate (11)

\[ \left( \frac{ds}{dt} \right)^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} \Rightarrow \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2 \]

: \( \mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \)

\[ \Rightarrow (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 \Rightarrow d\mathbf{r} \cdot d\mathbf{r} \]

: \( d\mathbf{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} \)

\( ds \) is called the linear element of \( C \).

Arc Length as Parameter

The parameter \( t \) can be replaced by the arc length \( s \) in the equation of a curve.

For example, the unit tangent vector is given as

\[ \mathbf{u}(s) = \mathbf{r}'(s) = \frac{d\mathbf{r}}{ds} \]

where

\[ \mathbf{r}'(s) = \lim_{\Delta s \to 0} \frac{\mathbf{r}(s + \Delta s) - \mathbf{r}(s)}{\Delta s} \]

Note that

\[ \left| \mathbf{r}'(s) \right| = \frac{ds}{ds} = 1 \]

A helix

\[ \mathbf{r}(t) = [\cos t, \sin t, ct] \quad \rightarrow \quad \mathbf{r}'(t) = [-\sin t, \cos t, c] \]

\[ \mathbf{r}' \cdot \mathbf{r}'' = a^2 + c^2 \]

The arc length is

\[ s = \int_0^t \sqrt{a^2 + c^2} \, dt = t \sqrt{a^2 + c^2} \]

Replace \( t \) by \( s \) in \( \mathbf{r}'(t) \)

\[ \mathbf{r}' \left( \frac{s}{\sqrt{a^2 + c^2}} \right) = a \cos \left( \frac{s}{\sqrt{a^2 + b^2}} \right) \mathbf{i} + a \sin \left( \frac{s}{\sqrt{a^2 + b^2}} \right) \mathbf{j} + c - \frac{s}{\sqrt{a^2 + c^2}} \mathbf{k} \]

\[ = \mathbf{r}'(s) \], a new function

- For a circle, let \( c=0 \) then \( t = s / a \)

\[ \rightarrow \mathbf{r}' \left( \frac{s}{a} \right) = a \cos \left( \frac{s}{a} \right) \mathbf{i} + a \sin \left( \frac{s}{a} \right) \mathbf{j} \]

Curves in Mechanics.  Velocity.  Acceleration

A curve \( C \) may represent a path of a moving body.

\( \mathbf{r}(t) \) \quad : t is time in this case

\( \mathbf{v}(t) = \mathbf{r}'(t) \) \quad : velocity vector

The magnitude

\[ |\mathbf{v}| = |\mathbf{r}'| \quad \Rightarrow \quad \frac{d\mathbf{r}}{ds} \Rightarrow \frac{ds}{dt} \quad (\therefore \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \mathbf{u}(s) \frac{ds}{dt}) \]

Speed of the body along the curve,

\( \mathbf{v} \) is parallel to the tangent of \( C \).

\( \mathbf{r}'(t)' \) has direction of \( \mathbf{r}'(s)' \)

(\( \mathbf{v} \) is the velocity of the moving body along \( C \))

\[ \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \) \quad : acceleration vector

Tangential and Normal Acceleration

In general

\[ \mathbf{a} = \mathbf{a}_{\text{tan}} + \mathbf{a}_{\text{norm}} \] : tangential and normal acceleration vectors

Proof:

\[ \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} \Rightarrow \frac{d\mathbf{r}}{ds} \frac{ds}{dt} \quad : s, \text{arc length} \]

\[ = \mathbf{u}(s), \text{unit tangent vector on } C \]

Differentiate the both sides

\[ \mathbf{a}(t) = \frac{d}{dt} \left[ \mathbf{u}(s) \frac{ds}{dt} \right] = \frac{d\mathbf{u}}{ds} \left( \frac{ds}{dt} \right) + \mathbf{u}(s) \frac{d^2s}{dt^2} \quad : \mathbf{u}(s), \text{tangential vector.} \quad \frac{d\mathbf{u}(s)}{ds}, \text{normal vector} \]

Normal acceleration.  Tangential acceleration.
Example 7  Centripetal(구심) acceleration.  Centrifugal(원심) Force.

The revolving path of a small body
\[ \vec{r}(t) = R \cos(\omega t) \hat{i} + R \sin(\omega t) \hat{j} \]
\( : \omega \), angular speed

The velocity vector
\[ \vec{v} = \vec{r}'(t) = -R \omega \sin(\omega t) \hat{i} + R \omega \cos(\omega t) \hat{j} \]

The speed of the body
\[ |\vec{v}| = R \omega \]

Angular speed = Linear speed / Distance to the center
\( : \omega \)

The acceleration vector
\[ \vec{a} = \vec{v}' = -R \omega^2 \cos(\omega t) \hat{i} - R \omega^2 \sin(\omega t) \hat{j} = -\omega^2 \vec{r} \]

\[ \rightarrow \quad |\vec{a}| = \omega^2 |\vec{r}| \Rightarrow \omega^2 R \quad : \text{centripetal acceleration} \]

Direction toward the center

If the body is to be in the circular orbit during rotation, someone should supply the centripetal force to the body. Otherwise, it will deviate from the orbit due to the centrifugal force.

Example 8  Superposition of Rotations

A projectile moves along a meridian of the rotating earth. Find its acceleration.

The angular speed of the earth rotation is \( \omega \).

The unit vector \( \vec{b} \) also rotates,
\[ \vec{b}(t) = \cos(\omega t) \hat{i} + \sin(\omega t) \hat{j} \]

The projectile on the meridian with an angular speed of \( \gamma \). Its position vector,
\[ \vec{r}(t) = R \cos(\gamma t) \hat{b} + R \sin(\gamma t) \hat{k} \quad : R \), radius of the earth

The derivatives
\[ \vec{v} = \vec{r}'(t) = R \cos(\gamma t) \dot{\vec{b}} - \gamma R \sin(\gamma t) \hat{b} + \gamma R \cos(\gamma t) \hat{k} \]
\[ \vec{a} = \vec{v}' = R \cos(\gamma t) \ddot{\vec{b}} - 2 \gamma R \sin(\gamma t) \dot{\vec{b}} - \gamma^2 R \cos(\gamma t) \dot{\vec{b}} - \gamma^2 R \sin(\gamma t) \hat{k} \]

Inserting \( \dot{\vec{b}} = -w^2 \cos(w t) \hat{i} - w^2 \sin(w t) \hat{j} = -w^2 \vec{b} \),
\[ \ddot{\vec{b}} = -\omega^2 R \cos(\gamma t) \hat{b} - 2\gamma R \sin(\gamma t) \dot{\vec{b}} - \gamma^2 \vec{r} \]

Centripetal acceleration by the earth.
\[ \uparrow \quad \text{Centripetal acceleration by the rotation of } P \text{ on the meridian.} \]

Coriolis acceleration due to interaction of two rotations.

Coriolis acceleration is in the direction opposite to the earth’s rotation
\[ \rightarrow \quad \text{The projectile deviates from the meridian toward the earth’s rotation direction in the Northern Hemisphere. (??)} \]

If the projectile is to be in the meridian during rotation, someone should supply the Coriolis acceleration to the projectile. Otherwise, it will deviate from the meridian in the direction opposite to the Coriolis acceleration.

Curvature and Torsion (skip)
9.6 Calculus Review

**Chain Rule**

Let all the functions be continuous and have continuous first partial derivatives in their domains. Let every point \((u, v)\) in \(B\) has the corresponding point \([x(u, v), y(u, v), z(u, v)]\) in \(D\).

(u: age, v: weight, \([x, y, u]\): location of participant, w: prize money in climbing competition)

The function defined in \(B\),

\[w = f[x(u, v), y(u, v), z(u, v)]\]

has first partial derivatives

\[\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}\]

\[\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}\]

**Simple examples:**

If \(w = f(x, y, z)\) and \(x = x(t), y = y(t), z = z(t)\), then

\[\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}\]

If \(w = f(x)\) and \(x = x(t)\), then

\[\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t}\]

**Mean Value Theorem**

Let \(f(x, y, z)\) be continuous and have continuous first partial derivative in a domain \(D\). Let two points, \(P_0 : (x_0, y_0, z_0)\) and \(P : (x_0 + h, y_0 + k, z_0 + l)\), be in \(D\) and the line segment between the points be in \(D\).

Then,

\[f(x_0 + h, y_0 + k, z_0 + l) - f(x_0, y_0, z_0) = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} + l \frac{\partial f}{\partial z}\]

The partial derivatives are evaluated at a point on the line segment

**A simple example:**

For a function of two variables

\[f(x_0 + h, y_0 + k) - f(x_0, y_0) = h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\]
9.7 Gradient of a Scalar Field. Directional Derivative

**Definition 1** Gradient

The *gradient* of a scalar function \( f(x,y,z) \) is defined as

\[
\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}
\]

: a vector function

"del" operator is defined as

\[
\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}
\]

: differential operator

\[\rightarrow \quad \nabla f = \nabla f \]

**Directional Derivative**

**Definition 2** Directional Derivative

*Directional derivative* of \( f \) at \( P \) in the direction of \( \mathbf{b} \) is defined by

\[
D_{\mathbf{b}} f = \lim_{s \to 0} \frac{f(Q) - f(P)}{s} ,
\]

where \( Q \) is a variable point displaced from \( P \) by \((s\mathbf{b})\)

The line \( L \) in Cartesian Coordinates

\[
\mathbf{r}(s) = x(s) \hat{i} + y(s) \hat{j} + z(s) \hat{k}
\]

\[
\mathbf{r}'(s) = x'(s) \hat{i} + y'(s) \hat{j} + z'(s) \hat{k}
\]

: this is a unit vector

\[\Rightarrow \mathbf{b}\]

The function \( f \) on \( C \) in parametric representation

\[f(x(s), y(s), z(s))\]

The directional derivative

\[
\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} + \frac{\partial f}{\partial z} \frac{dz}{ds} = \left( \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) \cdot \left( \frac{\partial x}{\partial s} \hat{i} + \frac{\partial y}{\partial s} \hat{j} + \frac{\partial z}{\partial s} \hat{k} \right)
\]

\[\rightarrow \quad \frac{df}{ds} = (\nabla f) \cdot \mathbf{b}\]

**Example 1** Directional derivative

Find the directional derivative of \( f(x,y,z) = 2x^2 + 3y^2 + z^2 \) at the point \( P: (2,1,3) \) in the direction of \( \mathbf{a} = \hat{i} - 2\hat{k}\)

\[
\nabla f = 4x\hat{i} + 6y\hat{j} + 2z\hat{k}
\]

\[\rightarrow \quad (\nabla f)_{(2,1,3)} = 8\hat{i} + 6\hat{j} + 6\hat{k}\]

The unit direction vector

\[\hat{a} = \frac{1}{\sqrt{5}} \hat{i} - \frac{2}{\sqrt{5}} \hat{k}\]

The directional derivative

\[D_{\hat{a}} f = \left( \frac{1}{\sqrt{5}} \hat{i} - \frac{2}{\sqrt{5}} \hat{k} \right) \cdot (8\hat{i} + 6\hat{j} + 6\hat{k}) = -1.789\]
**Gradient Is a Vector. Maximum Increase**

**Theorem 1** Vector Character is Gradient.

For a scalar function \( f(P) = f(x, y, z) \), its gradient is a vector function and its direction corresponds to the maximum increase of \( f \) on \( P \).

Its magnitude and direction are independent of the coordinate systems.

proof:

\[
D_{\hat{b}} f = \hat{b} \cdot \nabla f \Rightarrow |\hat{b}| |\nabla f| \cos \gamma \Rightarrow |\nabla f| \cos \gamma
\]

\( \rightarrow \) The maximum directional derivative for \( \hat{b} \parallel \nabla f \), or \( \gamma = 0 \)

\( \nabla f \parallel \) direction of max. \( D_{\hat{b}} f \).

Its max. value is \( |\nabla f| \).

- The directional derivative

\[
\lim_{\Delta \to 0} \frac{f(Q) - f(P)}{\Delta s} : \text{independent of coordinates}
\]

and \( \hat{b} \) is fixed in space.

Therefore, \( \nabla f \) should be independent of a coordinate system.

**Gradient as Surface Normal Vector**

A surface \( S \):

\[
f(x, y, z) = c
\]

A curve \( C \):

\[
\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}
\]

Its derivative:

\[
\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}
\]

Tangent vector of the curve.

If \( C \) is on \( S \), the surface equation becomes

\[
f[x(t), y(t), z(t)] = c.
\]

Differentiate the surface function w.r.t \( t \)

\[
\frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial z} z' = 0 : x' = dx/dt, \ldots
\]

\( \rightarrow (\nabla f) \cdot \vec{r}' = 0 \)

\( \uparrow \) Tangent to \( C \) and \( S \).

Surface normal vector

**Theorem 2** Gradient as Surface Normal Vector

When a surface \( S \) is given by \( f(x, y, z) = c \),

\( \nabla f \) at a point \( P \) on \( S \) represents the normal vector of \( S \) at \( P \).

**Example 2** Gradient as a surface normal vector

A cone is given by \( z^2 = 4(x^2 + y^2) \). Find a unit surface normal vector at \( P : (1,0,2) \)

The surface function is \( 4(x^2 + y^2) - z^2 = 0 \)

\( \rightarrow \) \( \nabla f = 8x \hat{i} + 8y \hat{j} - 2z \hat{k} \)

\( \rightarrow \) \( \nabla f = 8\hat{i} - 4\hat{k} \)

The unit surface normal vector is \( \hat{n} = \frac{8}{\sqrt{80}} \hat{i} - \frac{4}{\sqrt{80}} \hat{k} \)

\( \text{Fig. 214. Gradient as surface normal vector} \)

\( \text{Fig. 215. Cone and unit normal vector n} \)
Vector Fields that are Gradients of Scalar Fields (Potentials)

A vector function can be obtained by the gradient of a scalar function.

$$\vec{v}(P) = \nabla f(P)$$

$f(P)$ is called potential.

In this case, the field by $\vec{v}(P)$ is a conservative field (no loss of energy).

9.8 Divergence of a Vector Field

A differentiable vector function

$$\vec{v}(x, y, z) = v_1(x, y, z)\hat{i} + v_2(x, y, z)\hat{j} + v_3(x, y, z)\hat{k}$$

The divergence of $\vec{v}$ is defined as

$$\text{div } \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Using del operator

$$\text{div } \vec{v} \Rightarrow \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}) = \nabla \cdot \vec{v} : \text{a scalar function}$$

**Theorem 1** Invariance of the divergence

<table>
<thead>
<tr>
<th>In $xyz$-space with $v_1, v_2, v_3$</th>
<th>In $x'y'z'$-space with $v'_1, v'_2, v'_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{div } \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$</td>
<td>$\text{div } \vec{v}' = \frac{\partial v'_1}{\partial x'} + \frac{\partial v'_2}{\partial y'} + \frac{\partial v'_3}{\partial z'}$</td>
</tr>
</tbody>
</table>

It will be proved in 10.7

- If $f$ is twice differentiable

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

Take the divergence

$$\text{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \Rightarrow \nabla^2 f : \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \text{Laplace operator}$$

$$\text{div}(\nabla f) = \nabla^2 f$$
Example 2  Flow of a Compressible fluid.  Physical Meaning of the Divergence

A small box $B$ with a volume $\Delta V = \Delta x \Delta y \Delta z$.  
No source or sink in the volume.
Fluid flows through the box $B$.

The fluid velocity vector
$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

**Flux density** is defined as
$$\vec{u} = p \vec{v} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k}$$

Density of the fluid

The flux density is the **transfer** of certain quantity across a unit area per unit time.

- Loss of mass by outward flow through surfaces of $B$
  (1) Loss of mass due to flow in $y$ direction during a time interval $\Delta t$
  $$\left(u_y\right)_{yz} \Delta x \Delta z \Delta t + \left[-u_y\right]_{y} \Delta x \Delta z \Delta t \Rightarrow \left[u_y\right]_{y} \Delta x \Delta z \Delta t \Rightarrow \Delta u_y \frac{\Delta V}{\Delta y} \Delta t$$
  $$\uparrow = \Delta u_y$$

  (2) Loss due to flow in $x$ direction during $\Delta t$
  $$\left(u_y\right)_{xz} \Delta y \Delta z \Delta t + \left[-u_x\right]_{x} \Delta y \Delta z \Delta t \Rightarrow \Delta u_x \frac{\Delta V}{\Delta x} \Delta t$$

  (3) Loss due to flow in $z$ direction during $\Delta t$
  $$\left(u_z\right)_{xy} \Delta x \Delta y \Delta t + \left[-u_z\right]_{z} \Delta x \Delta y \Delta t \Rightarrow \Delta u_z \frac{\Delta V}{\Delta z} \Delta t$$

The total loss of mass in $B$ during $\Delta t$
$$\left( \frac{\Delta u_x}{\Delta x} + \frac{\Delta u_y}{\Delta y} + \frac{\Delta u_z}{\Delta z} \right) \Delta V \Delta t = -\frac{\partial (p \Delta V)}{\partial t} \Delta t$$

$$\uparrow \quad \uparrow \quad \text{Reduced mass inside } B.$$  
Loss by flow through six side surfaces of $B$.

$$\nabla \cdot \vec{u} = -\frac{\partial p}{\partial t}$$

: **continuity equation**, conservation of mass

For a steady flow, $\frac{\partial p}{\partial t} = 0$  \quad \Rightarrow \quad \nabla \cdot \vec{u} = 0$

For a constant $\rho$ (incompressible)  \quad \Rightarrow \quad \nabla \cdot \vec{v} = 0$, **condition of incompressibility**
9.9 Curl of a Vector Field

The **curl** of a vector function \( \mathbf{v}(x, y, z) = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k} \) is defined as

\[
\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
v_x & v_y & v_z
\end{vmatrix}
\]

\[
\Rightarrow \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{i} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{j} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{k}
\]

**Example 1** Curl of a Vector Function

A vector function is given, \( \mathbf{v} = yz \mathbf{i} + 3zx \mathbf{j} + z \mathbf{k} \), find the curl of \( \mathbf{v} \)

\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
yz & 3zx & z
\end{vmatrix}
\]

\[
\Rightarrow -3 \mathbf{x} \mathbf{i} + y \mathbf{j} + 2z \mathbf{k}
\]

**Example 2** Rotation of a rigid body

The rotation is described by the angular speed vector \( \mathbf{\omega} \).

- Its direction : right hand rule
- Its magnitude : angular speed, \( \omega \)

The linear speed of a point on the body : \( \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \)

Let the axis of rotation be z axis, \( \mathbf{\omega} = \omega \mathbf{k} \)

\[
\Rightarrow \mathbf{v} = \mathbf{\omega} \times \mathbf{r} \Rightarrow 0 0 \omega \Rightarrow -y \mathbf{i} + x \mathbf{j}
\]

The curl of \( \mathbf{v} \)

\[
\text{curl } \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-\omega y & \omega x & 0
\end{vmatrix}
\]

\[
\Rightarrow 2 \omega \mathbf{k} \rightarrow \text{curl } \mathbf{v} = 2 \mathbf{\omega}
\]

**Theorem 2** Grad, Div, Curl

The curl of the gradient of a scalar function is always zero

\[
\nabla \times (\nabla f) = 0
\]

: irrotational

The divergence of the curl of a vector function is always zero

\[
\nabla \cdot (\nabla \times \mathbf{v}) = 0
\]

**Theorem 3** Invariance of the curl

\[
\text{curl } \mathbf{v} \text{ is a vector independent of coordinate systems}
\]