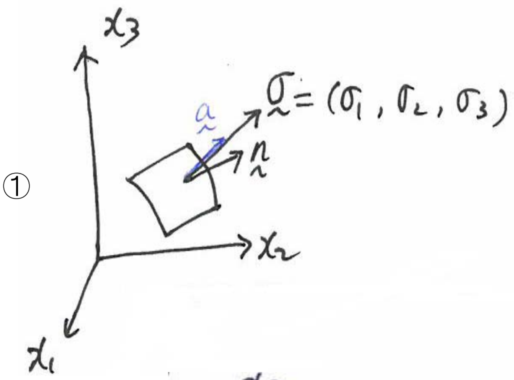
2.5 주전단응력과 최대전단응력

- 주응력축계 (x_1, x_2, x_3 축이 주응력 축으로 둠)에서
  수직응력은 주응력 $\sigma_I, \sigma_{II}, \sigma_{III}$ 이고 전단응력 성분은 모두 0.

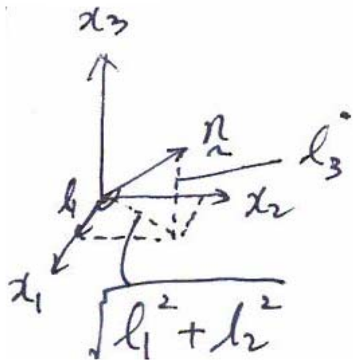
$\sigma_1 = \sigma_{11} l_1 + \sigma_{21} l_2 + \sigma_{31} l_3$ 에서 ($\sigma_{21} = 0$, $\sigma_{31} = 0$, $\sigma_{11} \rightarrow \sigma_I$)

$$\begin{pmatrix} \sigma_1 = \sigma_I l_1 \\ \sigma_2 = \sigma_{II} l_2 \\ \sigma_3 = \sigma_{III} l_3 \end{pmatrix}$$

로 둘 수 있다. —— ①



- 전단응력은 $\sigma_t = \sqrt{\sigma^2 - \sigma_n^2}$
  $= \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_n^2}$



$\therefore \sigma_t^2 (= \tau^2) = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_n^2$ $\quad \left\{ \begin{array}{l} \sigma = \sigma \tilde{a} \cdot \tilde{n} \\ = \sigma_1 \tilde{a}_1 \cdot \tilde{n} + \sigma_2 \tilde{a}_2 \cdot \tilde{n} + \sigma_3 \tilde{a}_3 \cdot \tilde{n} \end{array} \right.$ ($\tilde{a}_1 \cdot \tilde{n} = l_1$, $\tilde{a}_2 \cdot \tilde{n} = l_2$, $\tilde{a}_3 \cdot \tilde{n} = l_3$)

$l_1^2 + l_2^2 + l_3^2 = 1^2 = 1$

$(l_3^2 = 1 - l_1^2 - l_2^2)$ — ③

$= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1 l_1 + \sigma_2 l_2 + \sigma_3 l_3)^2$

대입

$= \sigma_I^2 l_1^2 + \sigma_{II}^2 l_2^2 + \sigma_{III}^2 l_3^2 - (\sigma_I l_1^2 + \sigma_{II} l_2^2 + \sigma_{III} l_3^2)$ —— ②

- l_1과 l_2로 미분한 값을 0으로 두면 ⇒ 전단응력의 특수값(최대, 최소, 변곡점)을 구함

주응력면에서 0

$let \ \dfrac{\partial \tau^2}{\partial l_1} = 0, \ \dfrac{\partial \tau^2}{\partial l_2} = 0$

→ 전단응력이 최대인 면 (방향여현)을 구하기 위함

$$l_1[(\sigma_I-\sigma_{III})l_1^2+(\sigma_{II}-\sigma_{III})l_2^2-\frac{1}{2}(\sigma_I-\sigma_{III})]=0$$ —— ④

$$l_2[(\sigma_I-\sigma_{III})l_1^2+(\sigma_{II}-\sigma_{III})l_2^2-\frac{1}{2}(\sigma_{II}-\sigma_{III})]=0$$

여기서, $l_1=l_2=0$, $l_{3=1}$이면, 전단응력이 0인 본래의 주응력면(전단응력=0)이므로

i) $l_1=0,\rightarrow l_2=\pm\frac{1}{\sqrt{2}},\ l_3=\pm\frac{1}{\sqrt{2}}$

ii) $l_2=0,\rightarrow l_1=\pm\frac{1}{\sqrt{2}},\ l_3=\pm\frac{1}{\sqrt{2}}$

iii) $l_1\neq 0,\ l_2\neq 0\rightarrow$ 괄호 안 모든 항을 0으로 두어야 함 ⇒ 해가 없음.

- 식 ③을 $l_2^2=1-l_1^2-l_3^2$ 을 대입해서 풀면,

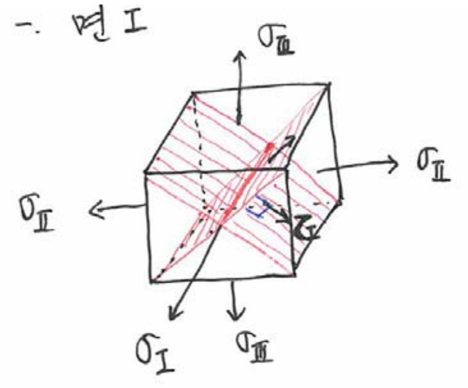
$$\therefore\ l_3=0,\rightarrow l_2=\pm\frac{1}{\sqrt{2}},\ l_1=\pm\frac{1}{\sqrt{2}}$$

- 이것을 정리하면,

면의 방향여현	면 I	면 II	면 III
l_1	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$
l_2	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$
l_3	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0

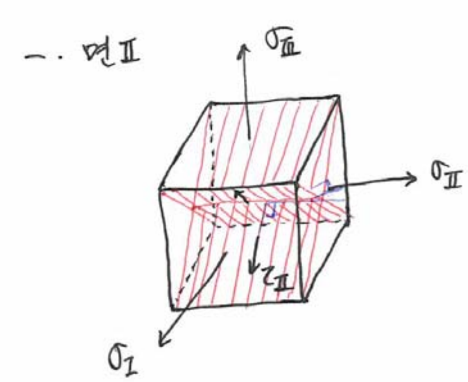
- 이것을 식 ②에 대입하여 주전단응력을 구하면,

$\tau_I=\frac{\sigma_{II}-\sigma_{III}}{2}$	$\tau_{II}=\frac{\sigma_I-\sigma_{III}}{2}$	$\tau_{III}=\frac{\sigma_I-\sigma_{II}}{2}$



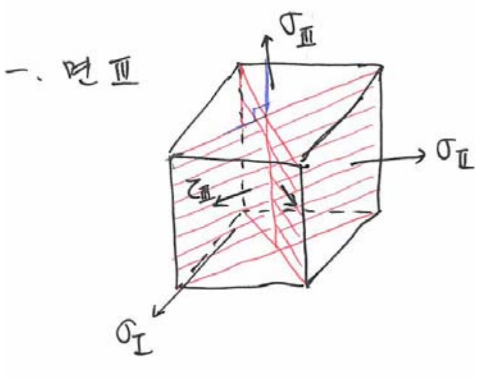
$$l_1 = 0 \leftarrow \tilde{a}_1 \bullet \tilde{n} = 0 \text{ (수직)}$$

$$\tau_I = \frac{\sigma_{II} - \sigma_{III}}{2}$$



$$l_2 = 0 \leftarrow \tilde{a}_2 \bullet \tilde{n} = 0 \text{ (수직)}$$

$$\tau_{II} = \frac{\sigma_I - \sigma_{III}}{2}$$



$$l_3 = 0 \leftarrow \tilde{a}_3 \bullet \tilde{n} = 0 \text{ (수직)}$$

$$\tau_{III} = \frac{\sigma_I - \sigma_{II}}{2}$$

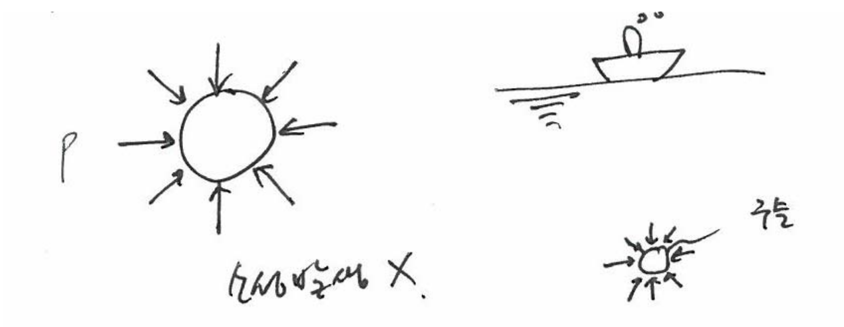
- 최대전단응력

일반적으로 $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$ 이므로,

$$\therefore\ \tau_{II} = \frac{\sigma_I - \sigma_{III}}{2} = \tau_{\max}$$

- $\sigma_I = \sigma_{II} = \sigma_{III}$ 이면, 어떠한 경사면에도 전단응력이 없음

⇒ 이러한 응력상태 : <u>등방향응력상태</u>



cf)

◎ $P = B(\frac{\triangle V}{V})$ (B : Bulk modulus)

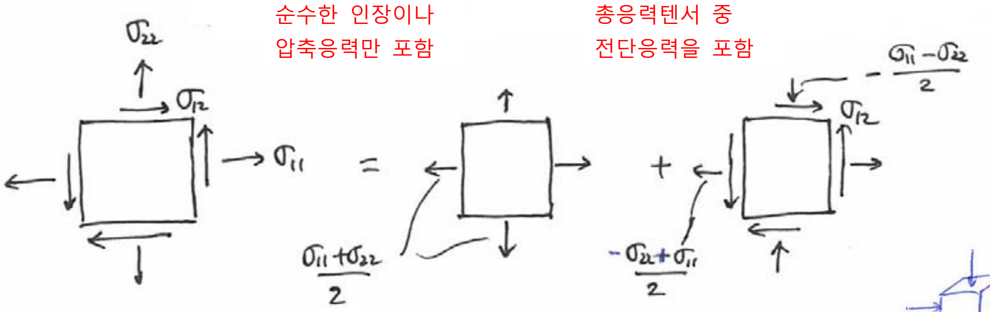
$\ast$ $B = \frac{E}{3(1-2\nu)}$ 관계를 보여라

2.7 편차응력

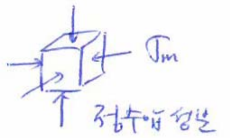
- 응력텐서는 평균(등방향)응력 σ_m과 편차응력 σ_{ij}'로 나눌 수 있다.

순수한 인장이나 압축응력만 포함

총응력텐서 중 전단응력을 포함



총응력 = 등방향응력 + 편차응력



평균수직응력(mean normal stress), σ_m

$$* \ \sigma_m = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{1}{3}J_1$$

- 등방향응력

$$\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$: 인장 or 압축응력만 나타내는 특수한 형태

- 편차응력

$$\sigma_{ij}' = \begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix} = \sigma_{ij} - \sigma_m\delta_{ij}$$

Kronecker delta

$$\begin{pmatrix} i=j \rightarrow 1 \\ i \neq j \rightarrow 0 \end{pmatrix}$$

- (증명) 편차응력은 전단응력을 포함한다.

Sol)

$$\sigma_I' = \sigma_I - \sigma_m = \sigma_I - \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III}) \quad : \text{주응력계에서}$$

$$= \frac{2\sigma_I - \sigma_{II} - \sigma_{III}}{3}$$

$$= \frac{(\sigma_I - \sigma_{II}) + (\sigma_I - \sigma_{III})}{3} = \frac{1}{3}(2\tau_{III} + 2\tau_{II}) \quad (\because \tau_{III} = \frac{\sigma_I - \sigma_{II}}{2})$$

$$= \frac{2}{3}(\tau_{III} + \tau_{II})$$

⇒ 편차응력은 전단응력을 포함하고 있으므로
소성변형을 일으키는데 매우 중요

- (증명) $\sigma_{11}' + \sigma_{22}' + \sigma_{33}' = 0$

$$\sigma_{11}' = \sigma_{11} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

$$\sigma_{22}' = \sigma_{22} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

$$+ \big) \quad \sigma_{33}' = \sigma_{33} - \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

$$\sigma_{11}' + \sigma_{22}' + \sigma_{33}' = \sigma_{11} + \sigma_{22} + \sigma_{33} - \frac{1}{3}(3\sigma_{11} + 3\sigma_{22} + 3\sigma_{33})$$

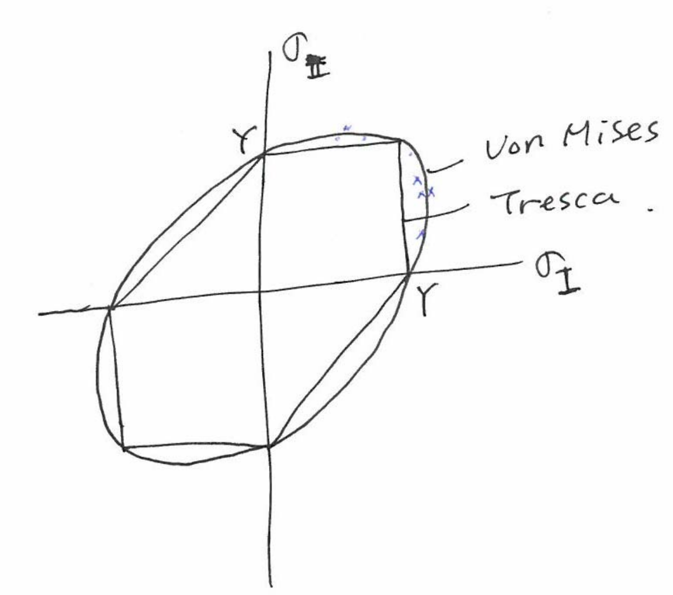
$$= 0$$

※ Criteria for Initial Yielding

- 현재까지 응력성분 사이의 어떠한 관계가 3차원 응력 상태 하에서의 항복을 단축 인장 시험에서 얻은 항복과 상관시켜 주는 이론은 없음.

- 비교적 잘 맞은 경험식 두 가지

  : Von Mises 항복 조건식, Tresca 항복 조건식



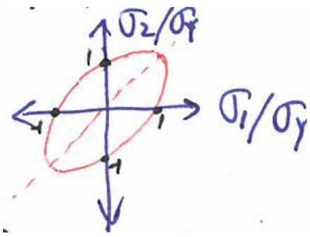
→ <u>기본 가정</u>

① 응력 상태는 주응력의 크기와 방향이 중요하지만 등방성 재료만으로 가정하여 <u>항복조건에서 주응력 크기만 기초</u>로 한다.

② 정수압 응력을 소성변형에 무관하기 때문에 주응력 자체보다는 <u>주응력 차의 절대값</u>에 기초를 둔다.

※ Von Mises 항복조건

- 편차응력의 2차 불변량 J_2'가 어떤 특정한 값 k^2 이하에서는 탄성 상태이고 J_2'가 k^2 이상이 되면 항복상태가 된다.



$$J_2 = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$

$$J_2' = -(\sigma_{11}'\sigma_{22}' + \sigma_{22}'\sigma_{33}' + \sigma_{33}'\sigma_{11}') + \sigma_{12}'^2 + \sigma_{23}'^2 + \sigma_{31}'^2$$

$$= -[\overset{0}{(\sigma_{11}' + \sigma_{22}' + \sigma_{33}')^2} - (\sigma_{11}'^2 + \sigma_{22}'^2 + \sigma_{33}'^2)] + \sigma_{12}'^2 + \sigma_{23}'^2 + \sigma_{31}'^2 \quad \text{—— ①}$$

here, $\sigma_{11}' + \sigma_{22}' + \sigma_{33}' = 0$ 이므로,

$$J_2' = \frac{1}{2}(\sigma_{11}' + \sigma_{22}' + \sigma_{33}')^2 + \sigma_{12}'^2 + \sigma_{23}'^2 + \sigma_{31}'^2 \quad \text{—— ②}$$

here, $\sigma_{11}'^2 + \sigma_{22}'^2 + \sigma_{33}'^2 = \frac{1}{3}[(\sigma_{11}' - \sigma_{22}')^2 + (\sigma_{22}' - \sigma_{33}')^2 + (\sigma_{33}' - \sigma_{11}')^2]$

$$\therefore J_2' = \frac{1}{6}[(\sigma_{11}' - \sigma_{22}')^2 + (\sigma_{22}' - \sigma_{33}')^2 + (\sigma_{33}' - \sigma_{11}')^2] + \sigma_{12}'^2 + \sigma_{23}'^2 + \sigma_{31}'^2 \quad \text{— ③}$$

$$\therefore J_2' = \frac{1}{6}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \quad \text{—— ④}$$

$$(\sigma_{11}' + \sigma_{22}' + \sigma_{33}')^2 = 0 = \sigma_{11}'^2 + \sigma_{22}'^2 + \sigma_{33}'^2 + 2(\sigma_{11}'\sigma_{22}' + \sigma_{22}'\sigma_{33}' + \sigma_{33}'\sigma_{11}')$$

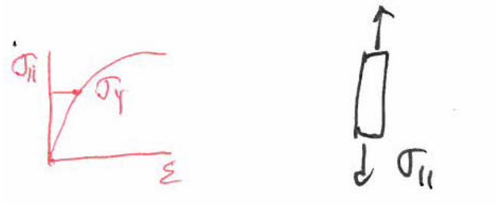
$$\therefore \sigma_{11}'^2 + \sigma_{22}'^2 + \sigma_{33}'^2 = -2(\sigma_{11}'\sigma_{22}' + \sigma_{22}'\sigma_{33}' + \sigma_{33}'\sigma_{11}')$$
$$= \sigma_{11}'^2 - 2\sigma_{11}'\sigma_{22}' + \sigma_{22}'^2 + \sigma_{22}'^2 - 2\sigma_{22}'\sigma_{33}' + \sigma_{33}'^2 + \sigma_{33}'^2 - 2\sigma_{33}'\sigma_{11}' + \sigma_{11}'^2$$
$$- 2(\sigma_{11}'^2 + \sigma_{22}'^2 + \sigma_{33}'^2)$$

$$= (\sigma_{11}' - \sigma_{22}')^2 + (\sigma_{22}' - \sigma_{33}')^2 + (\sigma_{33}' - \sigma_{11}')^2 - 2(\sigma_{11}'^2 + \sigma_{22}'^2 + \sigma_{33}'^2)$$

$$\therefore 3(\sigma_{11}'^2 + \sigma_{22}'^2 + \sigma_{33}'^2) = (\sigma_{11}' - \sigma_{22}')^2 + (\sigma_{22}' - \sigma_{33}')^2 + (\sigma_{33}' - \sigma_{11}')^2$$

- 단축인장(σ_{11}만 존재)에서 얻은 σ_Y를 대입

$$J_2' = \frac{1}{6}(2\sigma_{11}^2) = \frac{1}{6}(2\sigma_Y^2) = k^2$$



$$\therefore k = \frac{1}{\sqrt{3}}\sigma_Y \qquad \text{—— ⑤}$$

- 따라서,

$$\frac{1}{6}[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2]+\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2 = {\sigma_Y}^2$$

$$\therefore \sigma_Y = \frac{1}{\sqrt{2}}[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)]^{\frac{1}{2}}$$

- 임의의 하중에서 σ_{ij}와 단축인장응력에서 구한 σ_Y (항복응력 비교)

$$\therefore \bar{\sigma} = \frac{1}{\sqrt{2}}[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)]^{\frac{1}{2}} \qquad \text{—— ⑥}$$

Called to
Von Mises Stress
(effective stress)