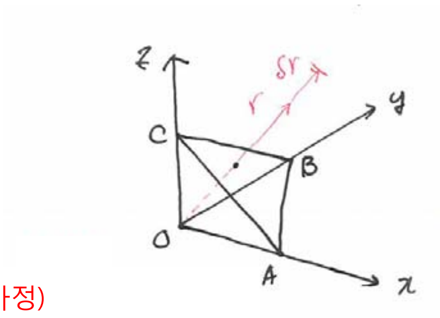
3.5 주변형률

- 주응력면 ~ 전단응력이 작용하지 않는 서로 직교하는 3개의 면 (3차원)
- 주변형률(principal strain)면 ~ 전단변형률이 발생하지 않으면, 신장이나 수축만 발생(방향은 변하지 않음, 회전x)



(가정)
$\triangle ABC$:주변형률면

방향은 주면 ABC와 동일(가정)
→ r와 δr의 O_x, O_y, O_z 방향의 성분은 비례

$$\epsilon = \frac{\delta r}{r} = (\epsilon_x = \frac{\delta u}{\delta x}) = (\epsilon_x = \frac{\delta v}{\delta y}) = (\epsilon_z = \frac{\delta w}{\delta z})$$

(ϵ : 주변형률)

$$\therefore \ \delta u = \epsilon \delta x \ , \ \delta v = \epsilon \delta y \ , \ \delta w = \epsilon \delta z \qquad \text{—— ①}$$

- 여기서, Taylor 정리에 의해 변위 성분 δu는 다음과 같다.

$$\delta u = \frac{\partial u}{\partial x}(=\epsilon_x)\delta x + \frac{\partial u}{\partial y}(=\epsilon_{xy})\delta y + \frac{\partial u}{\partial z}(=\epsilon_{xz})\delta z \ , \ u = f(x,y,z)$$

skip

* 일반적인 변위u_i식
$u_i = e_{ij}x_j$
$e_{ij} = \omega_{ij} + \epsilon_{ij}$
회전x라면
$\rightarrow e_{ij} = \epsilon_{ij}$
$\therefore u_i = \epsilon_{ij}x_j$
$\delta u_i = \epsilon_{ij}\delta x_j$

따라서,
$$\begin{pmatrix} \delta u = \epsilon_x \delta x + \epsilon_{xy}\delta y + \epsilon_{xz}\delta z \\ \delta v = \epsilon_{yx}\delta x + \epsilon_y \delta y + \epsilon_{yz}\delta z \\ \delta w = \epsilon_{zx}\delta x + \epsilon_{zy}\delta y + \epsilon_z \delta z \end{pmatrix} \qquad \text{—— ②}$$

- ①, ② 식에서

$$\begin{aligned} (\epsilon_x - \epsilon)\delta x + \epsilon_{xy}\delta y + \epsilon_{xz}\delta z = 0 \\ \epsilon_{yx}\delta x + (\epsilon_y - \epsilon)\delta y + \epsilon_{yz}\delta z = 0 \\ \epsilon_{zx}\delta x + \epsilon_{zy}\delta y + (\epsilon_z - \epsilon)\delta z = 0 \end{aligned} \qquad \text{—— ③}$$

$$\therefore \ \epsilon \delta x_i = \epsilon_{ij}\delta x_j \qquad (i = x,y,z \ j = x,y,z)$$

or $(\epsilon_{ij} - \delta_{ij}\epsilon)\delta x_j = 0$ $\qquad \delta_{ij}(Kronecker \ \delta)$

$i = j \rightarrow 1 \ , \ i \neq j \rightarrow 0$

from ③

$$\begin{bmatrix} (\epsilon_x - \epsilon) & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & (\epsilon_y - \epsilon) & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & (\epsilon_z - \epsilon) \end{bmatrix} \begin{Bmatrix} \delta x \\ \delta y \\ \delta z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- 해를 가지기 위한 조건

$$\begin{bmatrix} (\epsilon_x - \epsilon) & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & (\epsilon_y - \epsilon) & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & (\epsilon_z - \epsilon) \end{bmatrix} = 0 \qquad \text{—— ④}$$

- 식 ④를 전개하면

$$\epsilon^3 - (\epsilon_x + \epsilon_y + \epsilon_z)\epsilon^2 - (\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{zx}^2 - \epsilon_x\epsilon_y - \epsilon_y\epsilon_z - \epsilon_z\epsilon_x)\epsilon$$
$$- (\epsilon_x\epsilon_y\epsilon_z + 2\epsilon_{xy}\epsilon_{yz}\epsilon_{zx} - \epsilon_x\epsilon_{yz}^2 - \epsilon_y\epsilon_{zx}^2 - \epsilon_z\epsilon_{xy}^2) = 0$$

or

$$\epsilon^3 - J_1\epsilon^2 - J_2\epsilon - J_3 = 0 \qquad \text{—— ⑤}$$

여기서 $J_1,\ J_2,\ J_3$: 변형률 불변량(strain invariant)

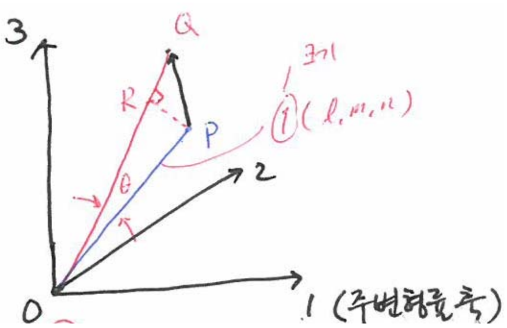
- 식 ⑤의 3개의 해 ⇒ 주변형률

그림 ABC가 주변형률 면이면, → $\epsilon_{xy} = \epsilon_{yz} = \epsilon_{zx} = 0$

$$\therefore\ J_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$$
$$J_2 = -(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1)$$
$$J_3 = \epsilon_1\epsilon_2\epsilon_3$$

3.6 주전단변형률과 최대전단변형률

- 전단응력이 최대가 되는 방향이 있듯이 전단변형률이 최대가 되는 방향 (= 주전단변형률 방향)



(변형 전)

OP : 크기 → 1(단위 길이)

방향여현 l, m, n

$(\overline{OP}^2 = l^2 + m^2 + n^2 = 1)$

↓

(변형 후)

OQ : 각 방향성분 $l(1+\epsilon_1)$, $m(1+\epsilon_2)$, $n(1+\epsilon_3)$

($\epsilon_{1,2,3}$: 주변형률)

① 길이방향 변형률 $\epsilon = \dfrac{\overline{RQ}}{\overline{OR}}$ 구하기

② 전단방향 변형률 $\theta \approx \dfrac{\overline{RP}}{\overline{OR}}$

①; - 따라서 변형 후 길이는

$$\overline{OQ}^2 = l^2(1+\epsilon_1)^2 + m^2(1+\epsilon_2)^2 + n^2(1+\epsilon_3)^2 \qquad \text{—— ①}$$

여기서, $\overline{OQ} = \overline{OR} + \overline{RQ}$ 이므로

$$(\overline{OR} + \overline{RQ})^2 = (\overline{OR}^2 + 2\overline{OR} \cdot \overline{RQ} + \overline{RQ}^2) \approx (\overline{OR}^2 + 2\overline{OR} \cdot \overline{RQ})$$

(변형량의 제곱을 0으로 가정)

$$\therefore \overline{OQ}^2 = l^2(1+2\epsilon_1) + m^2(1+2\epsilon_2) + n^2(1+2\epsilon_3)$$
$$= l^2 + m^2 + n^2 + 2(\epsilon_1 l^2 + \epsilon_2 m^2 + \epsilon_3 n^2)$$
$$= 1 + 2(\epsilon_1 l^2 + \epsilon_2 m^2 + \epsilon_3 n^2)$$

- $\overline{OR}^2 (\overline{OR} \approx \overline{OP} = 1)$으로 나누면

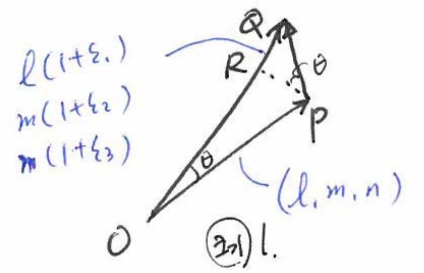
$$\frac{\overline{OQ}^2}{\overline{OR}^2} = 1 + 2\frac{\overline{RQ}}{\overline{OR}} \approx 1 + 2(\epsilon_1 l^2 + \epsilon_2 m^2 + \epsilon_3 n^2) \qquad \text{—— ②}$$

$$\therefore \epsilon \text{ (길이방향 변형률)} = \frac{\overline{RQ}}{\overline{OR}} = \epsilon_1 l^2 + \epsilon_2 m^2 + \epsilon_3 n^2 \qquad \text{—— ③}$$

②; - $\epsilon^2+\theta^2=\frac{\overline{RQ}^2}{\overline{OR}^2}+\frac{\overline{RP}^2}{\overline{OR}^2}=\frac{\overline{PQ}^2}{\overline{OR}^2}\approx\overline{PR}^2 \quad (\theta\approx 0)$

$$=1+2\frac{\overline{RQ}}{\overline{OR}}\approx 1+2(\epsilon_1 l^2+\epsilon_2 m^2+\epsilon_3 n^2)$$

여기서, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$,



$\ast$ $\overline{PQ}(=\overline{PR})$의 방향 : $\epsilon_1 l, \epsilon_2 m, \epsilon_3 n$

$\overline{OQ}$: $l(1+\epsilon_1), m(1+\epsilon_2), n(1+\epsilon_3)$

$\overline{OP}$: l, m, n

$\therefore\ \overline{PQ}=\overline{OQ}-\overline{OP}=l\epsilon_1, m\epsilon_2, n\epsilon_3$

- 따라서, $\epsilon^2+\theta^2\approx\epsilon_1^2 l^2+\epsilon_2^2 m^2+\epsilon_3^2 n^2$

$\therefore\ \theta^2\approx\epsilon_1^2 l^2+\epsilon_2^2 m^2+\epsilon_3^2 n^2-\epsilon^2$

$\approx\epsilon_1^2 l^2+\epsilon_2^2 m^2+\epsilon_3^2 n^2-(\epsilon_1 l^2+\epsilon_2 m^2+\epsilon_3 n^2)^2$

여기서 $l^2+m^2+n^2=1$ 이므로, $n^2=1-l^2-m^2$ —— $(\ast)$

$(\tan\theta\approx\theta)$ $\quad\therefore\ \theta^2\approx\epsilon_1^2 l^2+\epsilon_2^2 m^2+\epsilon_3^2(1-l^2-m^2)-(\epsilon_1 l^2+\epsilon_2 m^2+\epsilon_3 n^2)^2$ —— $(\ast\ast)$

- l과 m에 대하여 미분 후 → 0으로 두면,
  전단변형률이 최소 또는 최대가 되는 면의 방향여현을 구할 수 있음.

- $l[(\epsilon_1-\epsilon_3)l^2+(\epsilon_2-\epsilon_3)m^2-\frac{1}{2}(\epsilon_1-\epsilon_3)]=0$ —— ④

$m[(\epsilon_1-\epsilon_3)l^2+(\epsilon_2-\epsilon_3)m^2-\frac{1}{2}(\epsilon_2-\epsilon_3)]=0$

식 ④, $(\ast)$를 이용하면

l	0	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$
m	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$
n	± 1	0	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0

- 주전단변형률(principal shear strain)

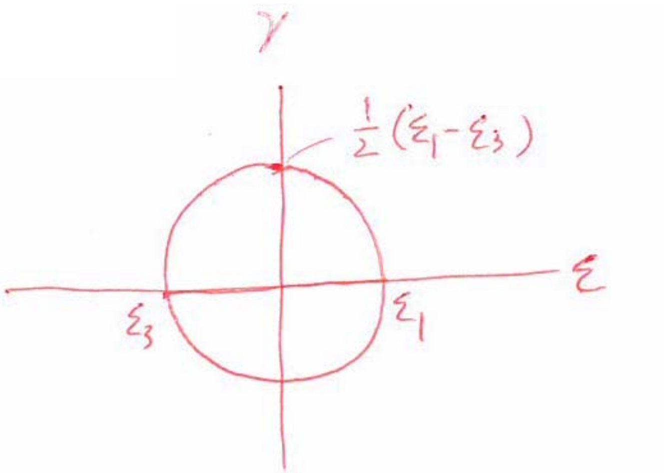
- 식 ④ 결과 및 식 (∗∗)

$$\gamma_1 = \pm \frac{1}{2}(\epsilon_2 - \epsilon_3)$$
$$\gamma_2 = \pm \frac{1}{2}(\epsilon_1 - \epsilon_3) \qquad \text{—— ⑤}$$
$$\gamma_3 = \pm \frac{1}{2}(\epsilon_1 - \epsilon_2)$$

- if $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$,

$$\gamma_{\max} = \pm \frac{1}{2}(\epsilon_1 - \epsilon_3)$$

(※ 여기서 $\gamma_{\max}$는 공칭전단변형률이 아님)

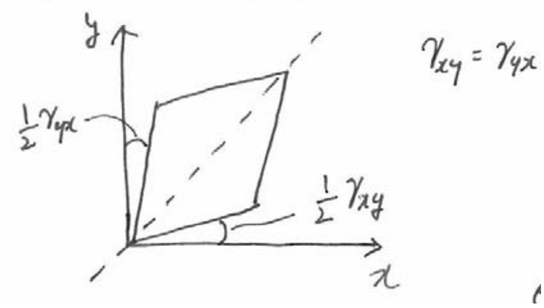


ϵ_{xy}(전단변형률) $= \frac{1}{2}\gamma_{xy}$(공칭전단변형률)

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

→ γ_{ij} : 회전각 ⇒ 공칭전단변형률

※ 변형률텐서 (변형텐서)



- strain tensor

$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$

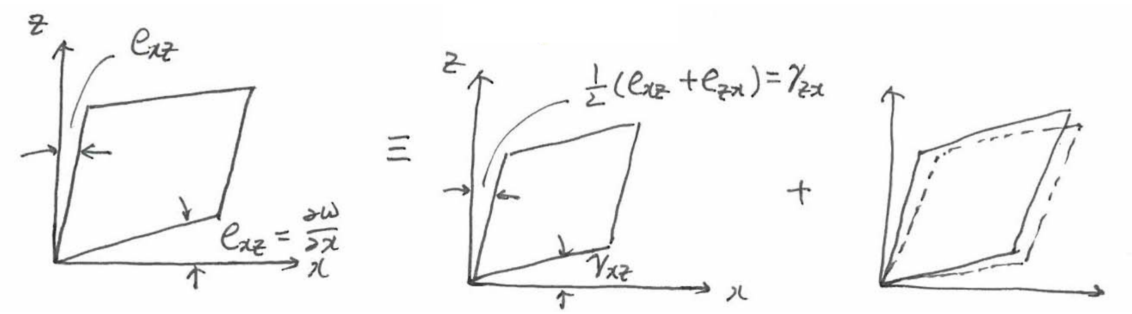
$\rightarrow \epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$

$e_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x_1} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$: 상대변형 텐서

ex)

$\epsilon_{xx} = \epsilon_x = \frac{1}{2}(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}) = \frac{1}{2}(e_{xx} + e_{xx}) = e_{xx}$

- $\epsilon_{ij} = \frac{1}{2}(e_{ij} + e_{ji})$, ω(회전텐서) $= \frac{1}{2}(e_{ij} - e_{ji})$ 로 두면,



변형 전단변형률 강체회전

∴ e_{ij}(상대 변형) $= \epsilon_{ij}$(전단변형)$+ \omega_{ij}$(강체회전)

ex) $e_{xy} = \epsilon_{xy} + \omega_{xy}$

$= \frac{1}{2}(e_{yx} + e_{xy}) + \frac{1}{2}(e_{yx} - e_{xy})$

$= e_{yx}$