경제수학 제 5 장

미분

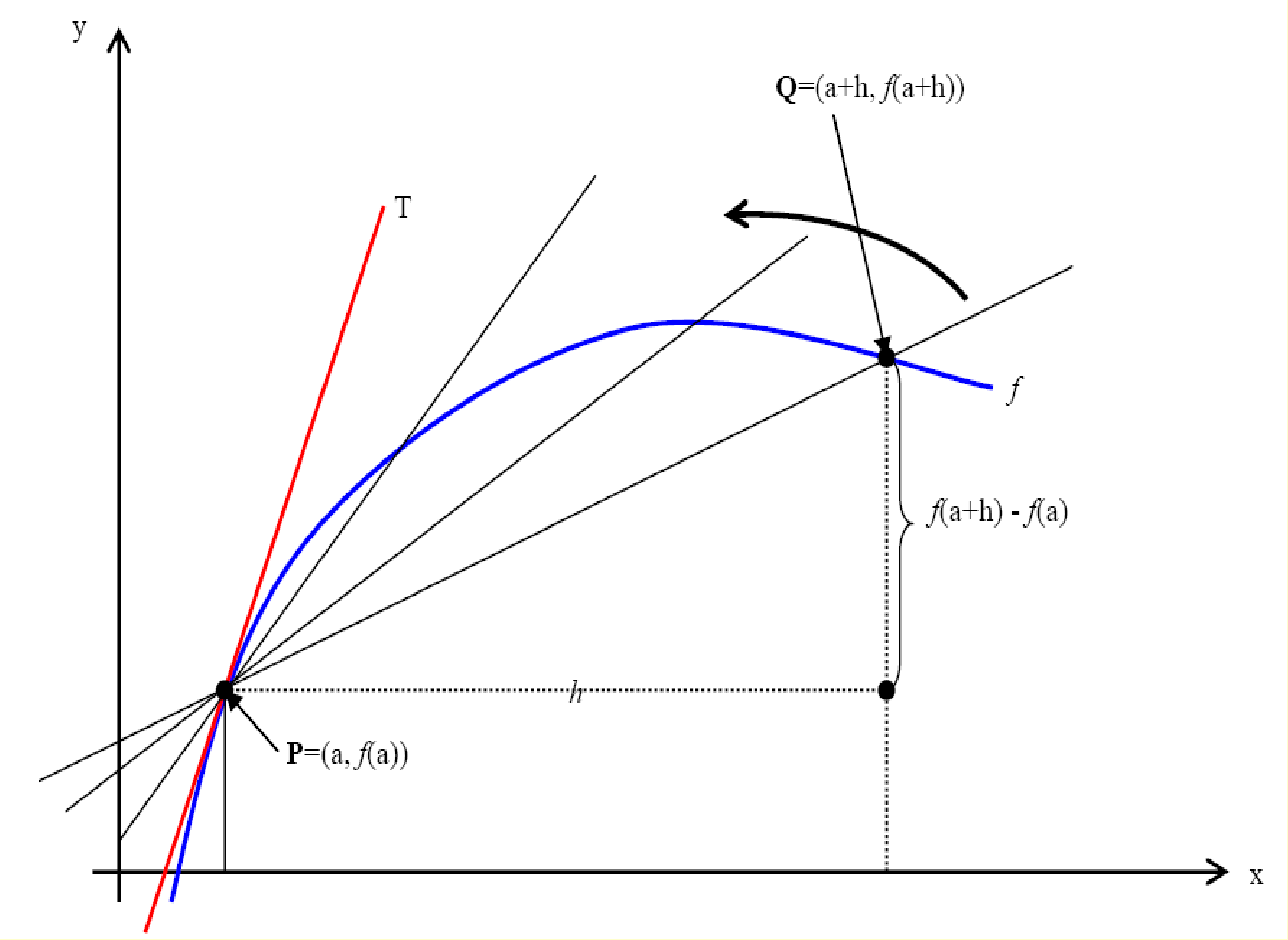
微分 (Differentiation)

- Differentiation provides a technique of measuring the rate at which one variable alters in response to changes in another

Changes for a Linear Function

- For a linear function
- The rate of change of y with respect to x is measured by
- The slope of the line =

$$\frac{\Delta y}{\Delta x} = \frac{(\text{distance up})}{(\text{distance to the right})}$$


y
Q=(a+h, f(a+h))
T
f
f(a+h) - f(a)
h
P=(a, f(a))
x

미분법 (Differentiation Terminology)

- Differentiation: finding the derivative (導函數) of a function
- Tangent (接線): a line that just touches a curve at a point
- Derivative of a function: the rate at which a function is changing with respect to an independent variable, measured at any point on the function by the slope of the tangent to the function at that point

Derivatives

- The derivative of y with respect to x is denoted $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

- The expression $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

  should be regarded as a single symbol and you should not try to work separately with parts of it

Using Derivatives

- The derivative $\frac{dy}{dx}$

  is an expression that measures the slope of the tangent to the curve at any point on the function $y = f(x)$
- A derivative measures the rate of change of y with respect to x and can only be found for smooth curves
- To be differentiable, a function must be continuous in the relevant range

The number $f'(a)$ gives the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$. The equation for a straight line passing through $(a, f(a))$ and having a slope $f'(a)$ is as follows:

$$y - f(a) = f'(a)(x - a) \qquad (2)$$

Ex) Use (1) to compute $f'(a)$ when $f(x) = x^2$. Find in particular $f'(1/2)$ and $f'(-1)$. Give geometric interpretations, and find the equation for the tangent at the point (1/2, 1/4).

Solution) $\dfrac{f(a+h) - f(a)}{h} = \dfrac{a^2 + 2ah + h^2 - a^2}{h} = \dfrac{2ah + h^2}{h} = 2a + h$. As h tends to 0, so $2a + h$ obviously tends to 2a. Thus, we can write $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{h\to 0}(2a + h) = 2a$. This shows that when $f(x) = x^2$, then $f'(a) = 2a$. For $a = 1/2$, we obtain $f'(1/2) = 2\cdot(1/2) = 1$. Similarly, $f'(-1) = 2\cdot(-1) = -2$. The equation of the tangent at (1/2, 1/4) will be $y - (1/4) = 1(x - 1/2)$, $y = x - 1/4$. And the equation of the tangent at (-1, 1) will be $y - 1 = (-2)(x + 1)$ or $y = -2x - 1$.

Problems)

1. Let $f(x) = 4x^2$. Show that $f(5+h) - f(5) = 40h + 4h^2$. Using this result, find $f'(5)$.

2. Let $f(x) = 3x^2 + 2x - 1$. Show that for $h \neq 0$, $f(x+h) - f(x) = 6xh + 2h + 3h^2$.

   Find in particular $f'(0)$, $f'(-2)$, and $f'(3)$. Find the equation of the tangent at the point (0, -1).

3. Show that $f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2}$

   (Hint: $[f(x+h) - f(x)]/h = -1/x(x+h)$)

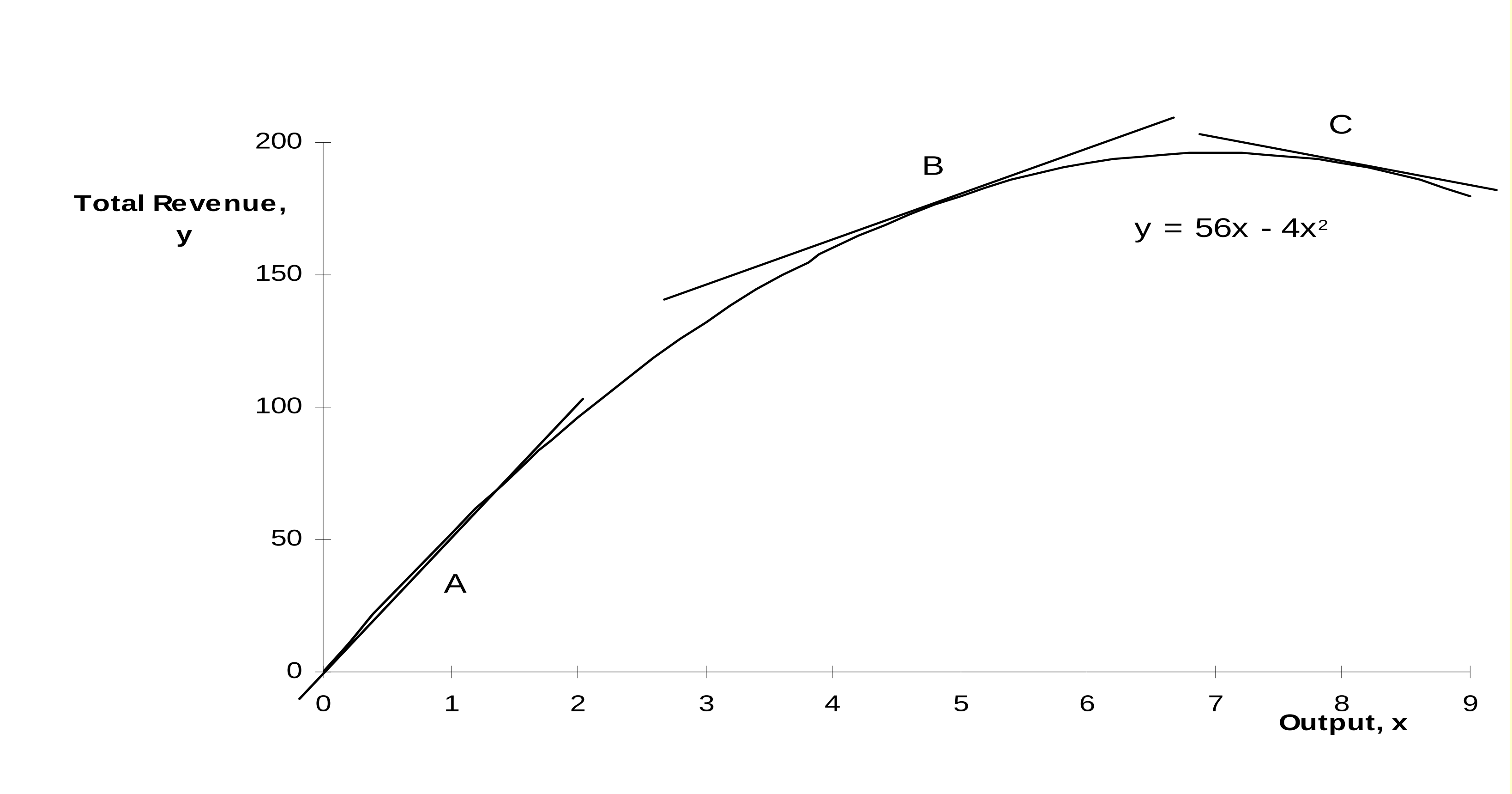
4. Find the slope for the tangent to the graph of *f* at the special points:

   (1) $f(x) = x^2 - 1$ at (1, 0) (2) $f(x) = \dfrac{3}{x} + 2$ at (3, 3) (3) $f(x) = x^3 - 2x$ at (0, 0)

   (4) $f(x) = x + \dfrac{1}{x}$ at (-1, -2) (5) $f(x) = x^4$ at (1, 1)

Tangents at points A, B and C

The slope of the tangent at A is steeper than that at B; the tangent at C has a negative slope



Small Increments Formula

- For small changes Δx it is approximately true that

$$\Delta y = \Delta x . \frac{\mathrm{d}y}{\mathrm{d}x}$$

- We can use this formula to predict the effect on y, Δy, of a small change in x, Δx
- This method is approximate and is valid only for small changes in x

Power-Function Rule

- If $y = ax^n$ where a and n are constants

$$\frac{dy}{dx} = nax^{n-1}$$

- Multiply by the power, then subtract 1 from the power

Constant Times a Function Rule

- Another way of handling the constant a in the function y = a.f(x) is to write it down as you begin differentiating and multiply it by the derivative of f(x)

$$\frac{\mathrm{d}\left(\mathrm{a}x^{\mathrm{n}}\right)}{\mathrm{d}x} = \mathrm{a}.\frac{\mathrm{d}\left(x^{\mathrm{n}}\right)}{\mathrm{d}x}$$

- The derivative of a constant times a function is the constant times the derivative of the function

Indices in Differentiation

- When differentiating power functions, remember the following from the rules of indices

$$x^1 = x$$

$$x^0 = 1$$

$$= x^{-n} \quad \frac{1}{x^n}$$

$$\sqrt{x} = x^{0.5} = x^{1/2}$$

Sum – Difference Rule

- If $y = \mathrm{f}(x) + \mathrm{g}(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}[\mathrm{f}(x)]}{\mathrm{d}x} + \frac{\mathrm{d}[\mathrm{g}(x)]}{\mathrm{d}x}$$

- If $y = \mathrm{f}(x) - \mathrm{g}(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}[\mathrm{f}(x)]}{\mathrm{d}x} - \frac{\mathrm{d}[\mathrm{g}(x)]}{\mathrm{d}x}$$

- The derivative of a sum (difference) is the sum (difference) of the derivatives

Inverse Function Rule

- To find dy/dx, we may obtain dx/dy and turn it upside down, i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$$

- There must be just one y value corresponding to each x value so that the inverse function exists

Revenue Functions

- To find marginal revenue, MR, differentiate total revenue, TR, with respect to quantity, Q

- If TR = f(Q)

- MR = $\dfrac{\mathrm{d}\,TR}{\mathrm{d}\,Q}$

Short-run Production Functions

- The marginal product of labor is found by differentiating the production function with respect to labor
- If output produced, Q, is a function of the quantity of labor employed, L, then
- $Q = \mathrm{f}(L)$
- $\mathrm{MPL} = \dfrac{\mathrm{d}Q}{\mathrm{d}L}$

Total and Marginal Cost

- Marginal cost is the derivative of total cost, TC, with respect to Q, the quantity of output, i.e.
- $\mathrm{MC} = \dfrac{\mathrm{dTC}}{\mathrm{d}\,Q}$
- When MC is falling, TC bends downwards When MC is rising, TC bends upwards

Variable and Marginal Cost

- Marginal cost is also the derivative of variable cost, VC, with respect to Q, i.e.

- $\mathrm{MC} = \dfrac{\mathrm{dVC}}{\mathrm{d}Q}$

Point Elasticity of Demand and of Supply

- Point price elasticity = $\frac{\mathrm{d}Q}{\mathrm{d}P} \times \frac{P}{Q}$
- For price elasticity of demand use the equation for the demand curve
- Differentiate it to find $\mathrm{d}Q/\mathrm{d}P$ then substitute as appropriate
- Supply elasticity is found from the supply equation in a similar way

Finding Point Elasticities

- Point price elasticity = $\dfrac{\mathrm{d}Q}{\mathrm{d}P} \times \dfrac{P}{Q}$
- If the demand or supply function is given in the form $P = \mathrm{f}(Q)$, use the inverse function rule

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{1}{\mathrm{d}P/\mathrm{d}Q}$$

- For downward sloping demand curves, $\mathrm{d}Q/\mathrm{d}P$ is negative, so point elasticity is negative
  - ➢ as price falls the quantity demanded increases

Elasticity Values

- Demand elasticities are negative, but we ignore the negative sign in discussion of their size
- As you move along a demand or supply curve, elasticity usually changes
- Functions with constant elasticity:
  - Demand: $Q = \mathrm{k}/P$ where k is a constant
    has E = – 1 at all prices
  - Supply: $Q = \mathrm{k}P$ where k is a constant
    has E = 1 at all prices