경제수학 제 8 장

편미분

편미분 (Partial Differentiation)

- The technique of partial differentiation lets us separately investigate the impact on the dependent variable as any one of the independent variables changes
- Partial derivative: measures the rate at which the dependent variable is changing as one independent variable changes, while all the other independent variables are held constant

Partial Derivatives

- For the general multivariate function

$$y = \mathrm{f}(w, x, z)$$

  we define three partial derivatives:

$$\frac{\partial y}{\partial w}, \frac{\partial y}{\partial x} \text{ and } \frac{\partial y}{\partial z}$$

- The curly ∂ symbol used in the notation indicates a partial derivative
- The symbol $\frac{\partial y}{\partial w}$ is read as 'the partial derivative of y with respect to w'

Finding Partial Derivatives

- We choose one variable, say w, whose effect we want to investigate and differentiate with respect to it, treating all the other independent variables as if they were constants
- Similarly we can find the other partial derivatives
- Each partial derivative is obtained by temporarily holding the other independent variables constant

Values of the Partial Derivatives

- The various partial derivatives of a function are usually different from one another
- Each may contain any or all of the independent variables in the original function
- The expressions can be evaluated for any values of the independent variables to give a measure of the corresponding rate of change in the dependent variable

Small Increments Formula

- The small increments formula for small changes Δx and Δz is

$$\Delta y = \frac{\partial y}{\partial x}\Delta x + \frac{\partial y}{\partial z}\Delta z$$

- This formula gives a useful approximation for small changes in x and z
- For linear functions the formula is valid for changes of any size in the relevant variables

Multipliers

- A multiplier is:
  - change in total income divided by the initial exogenous change which caused it
  - found by obtaining an expression for equilibrium income and partially differentiating

Examples of Multipliers

- Investment multiplier = $\frac{\partial Y}{\partial I}$
- Government expenditure multiplier = $\frac{\partial Y}{\partial G}$
- Lump sum direct taxation multiplier = $\frac{\partial Y}{\partial T}$
- Tax rate multiplier = $\frac{\partial Y}{\partial t}$

Predicting the Impact of a Change

- The small increments formula lets us predict the change in Y, ΔY, resulting from another specified change
- For example, if G changes by ΔG, the change in Y is given by

$$\Delta Y = \frac{\partial Y}{\partial G} . \Delta G$$

Differentials and Implicit Differentiation

- If $u = \mathrm{f}(x, y)$ the differentials form of the small increments formula is

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$$

- Implicit differentiation of $u = \mathrm{f}(x, y)$ where u is a constant gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y}$$

Further Implicit Differentiation

- if $u = \mathrm{f}(x,y,z)$

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y + \frac{\partial u}{\partial \mathrm{z}}\mathrm{dz}$$

- To find the partial derivative of any independent variable with respect to another
  - ➢ set du = 0
  - ➢ set the differential of the third independent variable equal to 0 and rearrange the equation

Demand Elasticities for good Y

- Own-price elasticity = $\frac{\partial Q_Y}{\partial P_Y} \cdot \frac{P_Y}{Q_Y}$

  (negative unless demand is upward sloping)

- Cross price elasticity = $\frac{\partial Q_Y}{\partial P_X} \cdot \frac{P_X}{Q_Y}$

  (positive for substitutes, negative for complements)

- Income elasticity of demand = $\frac{\partial Q_Y}{\partial M} \cdot \frac{M}{Q_Y}$

  (positive for normal goods, negative for inferior goods)

Logarithmic Demand Functions

- When the demand function for good Y has the log linear form
- $\ln Q_y = -\,\mathrm{a} - \mathrm{b} \ln P_y + \mathrm{c} \ln P_x + \mathrm{h} \ln M$
- The values of the own-price, cross-price and income elasticities are the coefficients of the relevant variable, namely – b, c and h

Long-run Production Functions

- For a long run production function

$$Q = \mathrm{f}(L,K)$$

$$\mathrm{MPL} = \frac{\partial Q}{\partial L}, \quad \mathrm{MPK} = \frac{\partial Q}{\partial K}$$

Homogeneity and Returns to Scale

- To show the returns to scale of a production function:
- Examine how output changes when all the inputs are changed in the same proportion
- A production function which is homogeneous
  - of degree 1 has constant returns to scale
  - of degree > 1 has increasing returns to scale
  - of degree < 1 has decreasing returns to scale

Euler's theorem

- Euler's theorem is a general result for homogeneous functions of degree *n*
- Euler's theorem states: $L \cdot \frac{\partial Q}{\partial L} + K \cdot \frac{\partial Q}{\partial K} = n \cdot Q$
- If firms with production functions that are homogeneous of degree 1 operate in competitive factor and product markets, the total factor payments exactly exhaust the value of the output

Cobb Douglas Production Functions

- have the general form $Q = \mathrm{A}L^{\alpha}K^{\beta}$
- are homogenous
- The parameters of the function α and β are the production elasticities of labor and capital respectively

Second Order Partial Derivatives

- Second order partial derivatives are found by partially differentiating a function twice
- We have:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)$$

$$\frac{\partial^2 y}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial y}{\partial z}\right)$$

Second Order Cross Partial Derivatives

- When the second partial differentiation is with respect to a different variable from the first, cross-partial derivatives are obtained
- We have:

$$\frac{\partial^2 y}{\partial z \cdot \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial y}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 y}{\partial x \cdot \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial z}\right)$$

- These cross partial derivatives are equal for all the functions we use in economics