Chapter 11
AC Circuit Power Analysis

Instantaneous Power Product of the time-domain voltage and time-domain current associated with the element or network of interest.

$$p(t) = v(t)i(t)$$

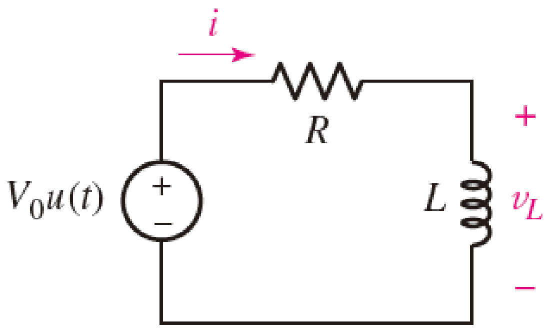
$$R: \ p(t) = v(t)i(t) = i^2(t)R = \frac{v^2(t)}{R}$$

$$L: p(t) = v(t)i(t) = i(t)L\frac{di(t)}{dt} = \left(\frac{1}{L}\int_{-\infty}^{t} v(t')dt'\right)v(t)$$

$$= \frac{1}{L}v(t)\int_{-\infty}^{t} v(t')dt' \quad \text{assume } v(-\infty) = 0$$

$$C: \ p(t) = v(t)i(t) = v(t)C\frac{dv(t)}{dt} = \left(\frac{1}{C}\int_{-\infty}^{t} i(t')dt'\right)i(t)$$

$$= \frac{1}{C}i(t)\int_{-\infty}^{t} i(t')dt' \quad \text{assume } i(-\infty) = 0$$



$$i(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)u(t)$$

$$p(t) = v(t)i(t) = \frac{V_0^2}{R}\left(1 - e^{-\frac{R}{L}t}\right)u^2(t)$$

Total power delivered by the source

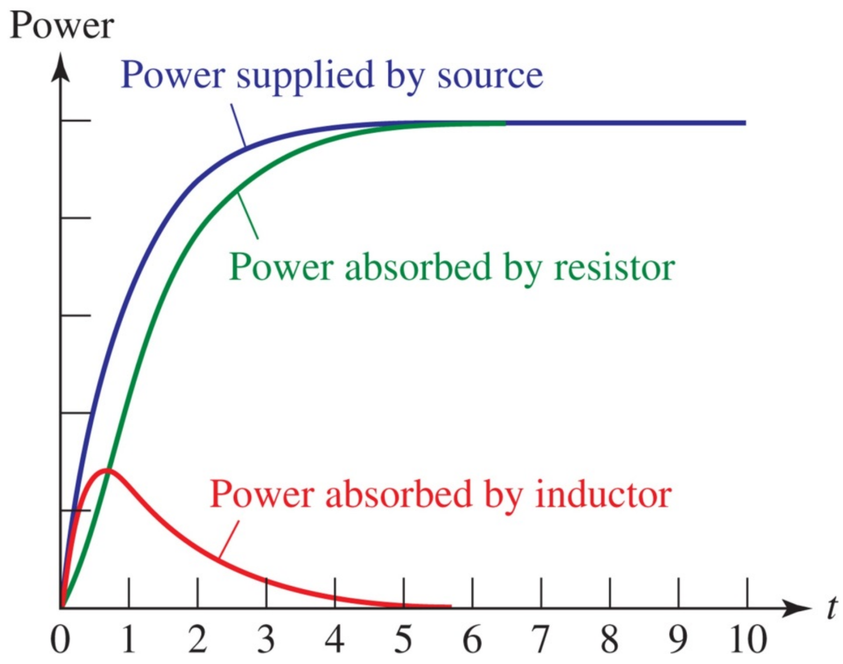
$$p_R(t) = i^2(t)R = R\frac{V_0^2}{R^2}\left(1 - e^{-\frac{R}{L}t}\right)^2 u^2(t)$$
$$= \frac{V_0^2}{R}\left(1 - e^{-\frac{R}{L}t}\right)^2 u(t)$$

Power delivered to the resistor

$$v_L(t) = L\frac{di(t)}{dt} = L\frac{V_0}{R}\left(\frac{R}{L}e^{-\frac{R}{L}t}\right)u(t)$$
$$+L\frac{V_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right)\delta(t) = V_0e^{-\frac{Rt}{L}}u(t)$$

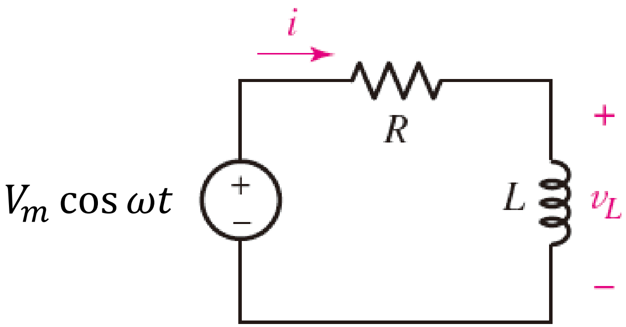
$$p_L(t) = v_L(t)i(t) = \frac{V_0^2}{R}e^{-\frac{Rt}{L}}\left(1 - e^{-\frac{R}{L}t}\right)u(t)$$

Power absorbed by the inductor



At all times *t*,
power supplied = power absorbed

Power due to Sinusoidal Excitation

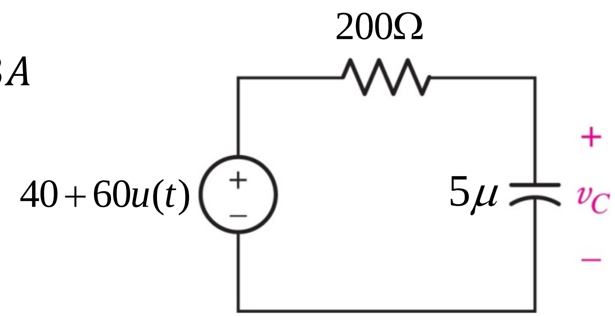


$$i(t) = I_m \cos(\omega t + \phi) \qquad I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \qquad \phi = -\tan^{-1}\frac{\omega L}{R}$$

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \phi)\cos\omega t$$

$$= \frac{V_m I_m}{2}\cos\phi + \frac{V_m I_m}{2}\cos(2\omega t + \phi)$$

Example 11.1 Find the power being absorbed by the capacitor and the resistor at t=1.2ms.



$$\tau = RC = 200 \times 5 \times 10^{-6} = 1\ ms, \quad v_C(0) = 40V,\ i(0) = 0.3A$$

$$\Rightarrow\ i(t) = 300e^{-t/\tau}\ mA,\ \ v_C(t) = (100 - 60^{-\frac{t}{\tau}})$$

$$p_R(t) = i^2(t)R = 300^2\left(e^{-\frac{t}{\tau}}\right)^2(200)$$

$$\Rightarrow p_R(t = 1.2\ m) = 300^2(e^{-1.2})^2(200) = 1.633\ W$$

$$p_C(t) = v_C(t)i(t)$$

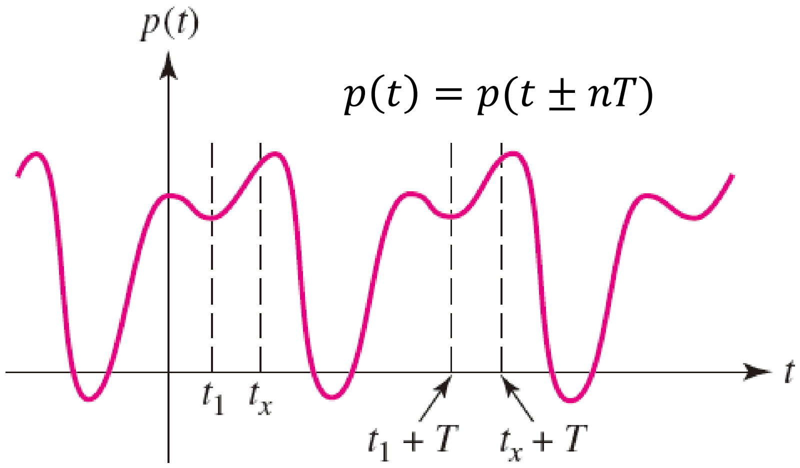
$$= \left(100 - 60^{-\frac{t}{\tau}}\right)\left(300e^{-\frac{t}{\tau}}\right)$$

$$p_C(t = 1.2\ m) = v_C(t = 1.2\ m)i(t = 1.2\ m) = (90.36)(81.93) = 7.403\ W$$

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t)dt$$

Average power selected on a general interval

Average power for periodic waveforms



$$p(t) = p(t \pm nT)$$

$$P_x = P = \frac{1}{T} \int_{t_x}^{t_x+T} p(t)dt$$

$$P = \frac{1}{nT} \int_{t_x}^{t_x+nT} p(t)dt\,, n = 1, 2, 3, \cdots$$

$$P = \frac{1}{nT} \int_{-nT/2}^{nT/2} p(t)dt \qquad P = \lim_{n\to\infty} \frac{1}{nT} \int_{-nT/2}^{nT/2} p(t)dt$$

Average power in the sinusoidal steady state

$$v(t) = V_m \cos(\omega t + \theta) \qquad i(t) = I_m \cos(\omega t + \phi)$$

$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta) \cos(\omega t + \phi)$$

$$= \frac{1}{2} V_m I_m \cos(\theta - \phi) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta + \phi)$$

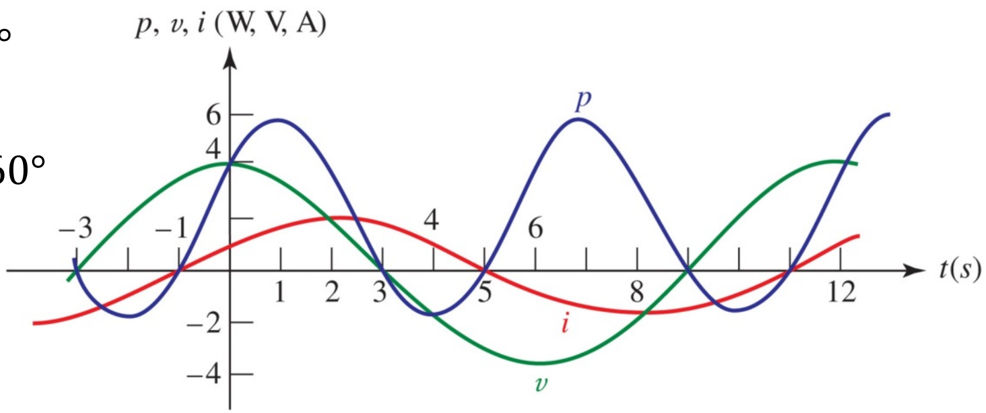
Average power

$$P = \frac{1}{2} V_m I_m \cos(\theta - \phi)$$

Example 11.2 Find both the average power and an expression for the instantaneous power.

$$v = 4\cos\left(\frac{\pi t}{6}\right) \Rightarrow \mathbf{V} = 4\angle 0° \qquad \mathbf{Z} = 2\ \angle 60°$$

Phasor current $\quad \mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{4\angle 0°}{2\angle 60°} = 2\angle -60°$



Average power

$$P = \frac{1}{2} V_m I_m \cos\big(0° - (-60°)\big) = 2\ W$$

$$p(t) = v(t)i(t) = \left(4\cos\left(\frac{\pi t}{6}\right)\right)\left(2\cos(\frac{\pi t}{6} - 60°)\right) = 2 + 4\cos(\frac{\pi t}{3} - 60°)\ W$$

Instantaneous power

- Average power absorbed by an ideal resistor

$$P_R = \frac{1}{2} V_m I_m \cos 0° = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2}\frac{V_m^2}{R} \qquad \cos(\theta - \phi) = \cos 0° = 1$$

Phase-angle difference is zero across a pure resistor

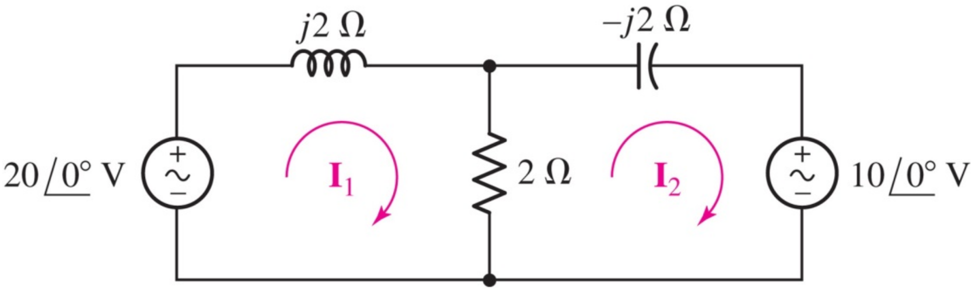
- Average power absorbed by purely reactive elements

Average power for (L , C)= 0 $\qquad \cos(\theta - \phi) = \cos \pm 90° = 0 \qquad P_X = 0$

Example 11.3 Find the average power being delivered to an impedance $\mathbf{Z}_L$ by $\mathbf{I}$.

$$Z_L = 8 - j11\,\Omega \qquad \mathrm{I} = 5\angle 20^\circ\ A \qquad \rightarrow\ P = \frac{1}{2} I_m^2 R = \frac{1}{2} 5^2 8 = 100\ W$$

Example 11.4 Find the average power absorbed by each of the three passive elements.



$$\mathbf{I}_1 = 5 - j10 = 11.18\angle -63.53^\circ$$

$$\mathbf{I}_2 = 5 - j5 = 7.071\angle -45^\circ$$

$$\mathbf{I}_1 - \mathbf{I}_2 = -j5 = 5\angle -90^\circ$$

$$P_{2\Omega} = \frac{1}{2} I_m^2 R = \frac{1}{2}(5)^2(2) = 25\ W$$

Absorb

$$P_{\text{left source}} = \frac{1}{2} V_m I_m \cos(\theta - \phi)$$

$$= \frac{1}{2}(20)(11.18)\cos(0-(-63.53)) = 50\ W$$

Deliver

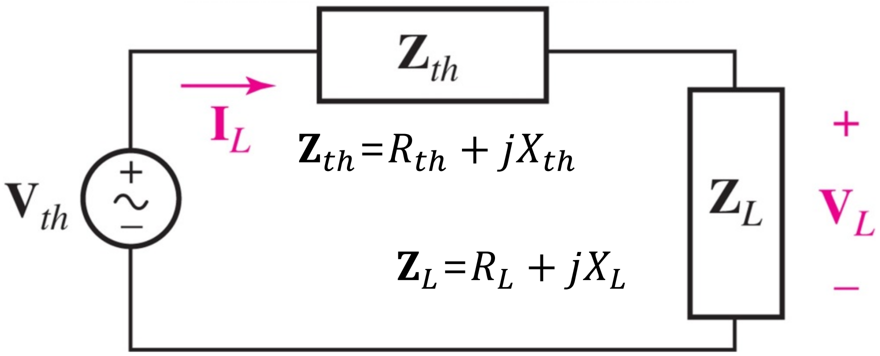
$$P_{\text{right source}} = \frac{1}{2} V_m I_m \cos(\theta - \phi)$$

$$= \frac{1}{2}(10)(7.071)\cos(0-(-45)) = 25\ W$$

Absorb

$$50 = 25 + 25$$

Maximum Power Transfer



An independent voltage source in *series with an impedance* Z_{th} delivers a maximum average power to that load impedance Z_L which is the conjugate of Z_{th}:

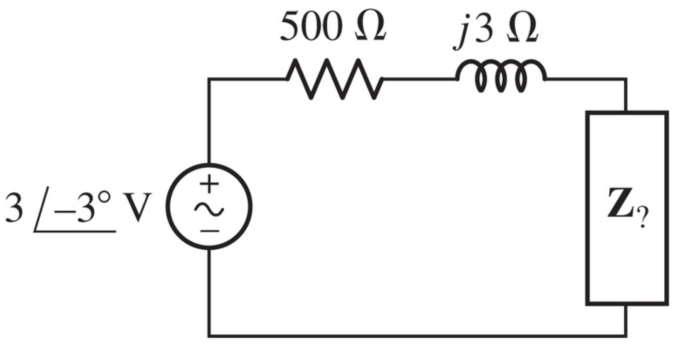
$$Z_L = Z_{th}^*$$

$$\mathbf{I}_L=\frac{\mathbf{V}_{th}}{\mathbf{Z}_{th}+\mathbf{Z}_L}=\frac{\mathbf{V}_{th}}{R_{th}+R_L+j(X_{th}+X_L)}$$

$$\mathbf{V}_L=\frac{\mathbf{Z}_L}{\mathbf{Z}_{th}+\mathbf{Z}_L}\mathbf{V}_{th}=\frac{R_L+jX_L}{R_{th}+R_L+j(X_{th}+X_L)}\mathbf{V}_{th}$$

$$P=\frac{1}{2}|\mathbf{V}_L||\mathbf{I}_L|\cos(\theta-\phi)=\frac{1}{2}\frac{|\mathbf{V}_{th}|^2\sqrt{R_L^2+X_L^2}}{(R_{th}+R_L)^2+(X_{th}+X_L)^2}\cos\left(\tan^{-1}\left(\frac{X_L}{R_L}\right)\right)$$

Example 11.5 Find Z for maximum power deliver. Source voltage is 3cos(100t-3°).



$$Z_? = Z_{Th}^* = (500+j3)^* = 500-j3\ \Omega$$

This impedance can be constructed by series connection of 500Ω resistor and 3.333mF capacitor.

$$\frac{1}{j\omega C}=-j\frac{1}{100\times 3.333\times 10^{-3}}=-j3$$

Average Power for Non-periodic Functions

Usually, the sum of periodic functions is periodic except the ratio of periods is irrational number.

$$i(t) = \sin t + \sin \pi t$$

$$\begin{aligned} i(t+2n\pi) &= \sin(t+2n\pi) + \sin \pi(t+2n\pi) \\ &= \sin t + \sin(\pi t + 2n\pi^2) \neq i(t) \end{aligned}$$ **Non-periodic**

$$i(t) = \sin t + \sin 3.14t$$

$$\begin{aligned} i(t+100\pi) &= \sin(t+100\pi) + \sin 3.14(t+100\pi) \\ &= \sin t + \sin(3.14t + 314\pi) = i(t) \end{aligned}$$ **Periodic with period** 100π

$$p(t) = i^2(t)R = i^2(t) \text{ with } R = 1$$

$$P = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} p(t)dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} (\sin^2 t + \sin^2 \pi t + 2\sin t \sin \pi t)dt = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

Generally

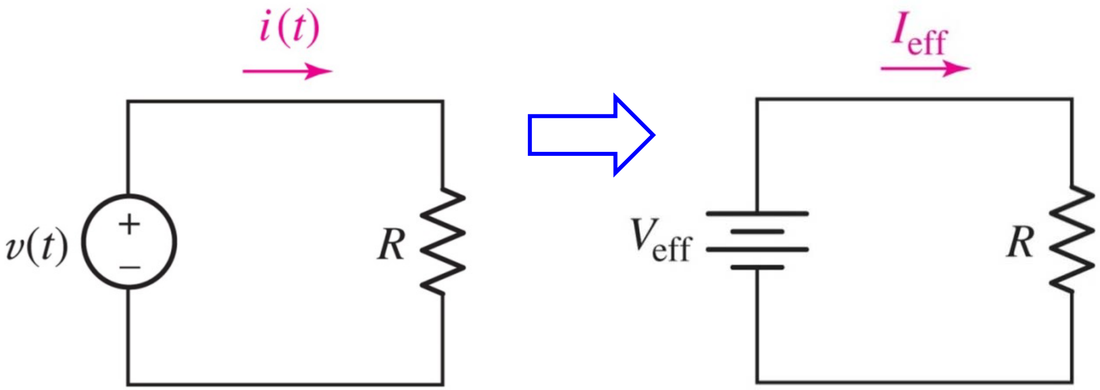
$$i(t) = I_{m_1}\cos\omega_1 t + I_{m_2}\cos\omega_2 t + \cdots + I_{m_N}\cos\omega_N t \qquad P = \frac{1}{2}\left(I_{m_1}^2 + I_{m_2}^2 + \cdots + I_{m_N}^2\right)R$$

Example 11.6 Find the average power delivered to a 4Ω resistor.

$$i_1(t) = 2\cos 10t - 3cos20t \quad A \qquad P = \frac{1}{2}(2^2 + 3^2)4 = 26\ W$$

Power outlet of 220 V do Not mean the instantaneous voltage of 220 V, but effective value of the sinusoidal voltage out.

The same power is delivered to the resistor in the circuits shown.



$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i^2 dt}$$

$$P = \frac{1}{T}\int_0^T i^2 R dt = R\left(\frac{1}{T}\int_0^T i^2 dt\right)$$

$$P = I_{eff}^2 R$$

The effective value is often referred to as the root-mean-square or RMS value.

Effective (RMS) value of a sinusoidal waveform

$$i(t) = I_m \cos(\omega t + \phi) \text{ with } T = \frac{2\pi}{\omega}$$

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2(\omega t + \phi)\, dt} = I_m \sqrt{\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\frac{1}{2} + \frac{1}{2} cos(2\omega t + 2\phi)\right] dt} = \frac{I_m}{\sqrt{2}}$$

Use of RMS values to compute Average Power

$$P = \frac{1}{2} I_m^2 R = \frac{1}{2}\left(\sqrt{2} I_{eff}\right)^2 R = I_{eff}^2 R \qquad P = \frac{V_{eff}^2}{R}$$

$$P = \frac{1}{2} V_m I_m \cos(\theta - \phi) = \frac{1}{2}\left(\sqrt{2} V_{eff}\right)\left(\sqrt{2} I_{eff}\right)\cos(\theta - \phi) \quad \therefore\ P = V_{eff} I_{eff} \cos(\theta - \phi)$$

Effective Value with Multiple-Frequency Circuits

$$i(t) = I_{m_1} \cos \omega_1 t + I_{m_2} \cos \omega_2 t + \cdots + I_{m_N} \cos \omega_N t \qquad \Rightarrow\ P = \frac{1}{2}\left(I_{m_1}^2 + I_{m_2}^2 + \cdots + I_{m_N}^2\right) R$$

$$P = \left(I_{eff_1}^2 + I_{eff_2}^2 + \cdots + I_{eff_N}^2\right) R \qquad \Rightarrow\ I_{eff} = \sqrt{I_{eff_1}^2 + I_{eff_2}^2 + \cdots + I_{eff_N}^2}$$

Let $v(t) = V_m \cos(\omega t + \theta)$, $i(t) = I_m \cos(\omega t + \phi)$ phase angle: $(\theta - \phi)$

Average power

$$P = \frac{1}{2} V_m I_m \cos(\theta - \phi) \qquad P = V_{eff} I_{eff} \cos(\theta - \phi)$$

Apparent power

$$P = V_{eff} I_{eff}$$ unit: VA

Power factor

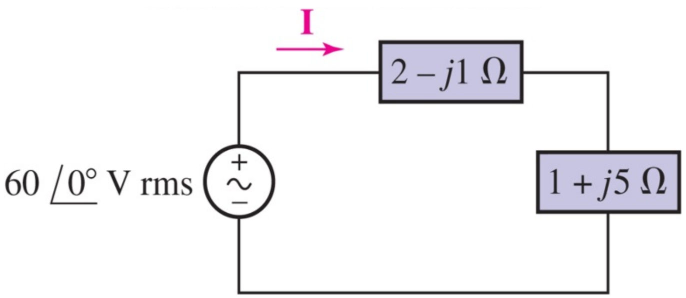
$$PF = \frac{averag\ power}{apparent\ power} = \frac{P}{V_{eff} I_{eff}}$$

Sinusoidal case

$$PF = \frac{P}{V_{eff} I_{eff}} = \cos(\theta - \phi)$$

pure resistive load: $PF = 1$
pure reactive load: $\theta - \phi = \pm 90° \rightarrow PF = 0$

Example 11.8 Calculate the average power delivered to each of the two impedances.



$$\mathbf{I} = \frac{60\angle 0°}{3 + j4} = 12\angle -53.13°\ A\ \ rms \qquad I_{eff} = 12$$

$$P_{upper} = I_{eff}^2 R_{top} = (12)^2(2) = 288\ W$$

$$P_{lower} = I_{eff}^2 R_{right} = (12)^2(1) = 144\ W$$

Apparent power: $V_{eff} I_{eff} = (60)(12) = 720\ VA$

Load: **inductive**

$$PF = \frac{P}{V_{eff} I_{eff}} = \frac{288 + 144}{720} = \frac{432}{720} = 0.6\ lagging$$

Define the complex power **S** as $$\mathbf{S} = \mathbf{V}_{eff}\mathbf{I}_{eff}^{*} = V_{eff} I_{eff} e^{j(\theta-\phi)} = P + jQ$$

- the real part of **S** is P, the average power
- the imaginary part of **S** is Q, the reactive power, which represents the flow of energy back and forth from the source (utility company) to the inductors and capacitors of the load (customer).

Average power

$$P = V_{eff} I_{eff} \cos(\theta - \phi) = V_{eff} I_{eff} Re\{e^{j(\theta-\phi)}\} = Re\{V_{eff} e^{j\theta} I_{eff} e^{-j\phi}\} = Re\{\mathbf{V}_{eff}\mathbf{I}_{eff}^{*}\}$$

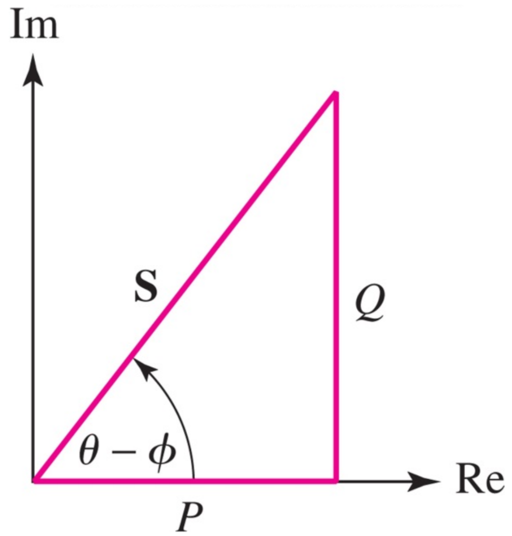
$Q = V_{eff} I_{eff} \sin(\theta - \phi)$ reactive power unit: VAR

$$\mathbf{I}_{eff}^{*} = I_{eff} e^{-j\phi}$$

TABLE 11.1 Summary of Quantities Related to Complex Power

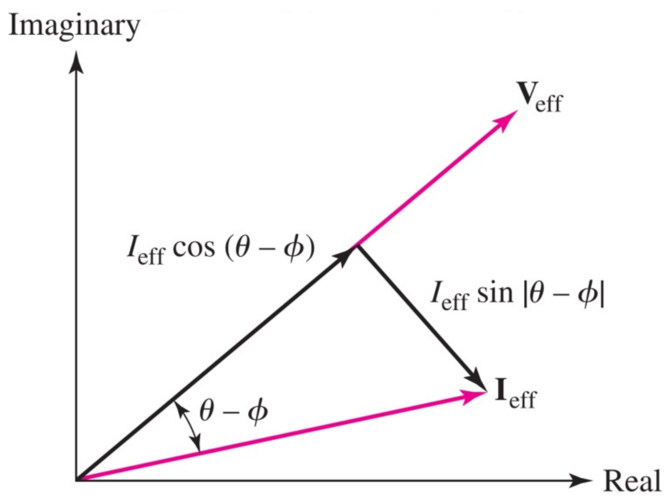
Quantity	Symbol	Formula	Units
Average power	P	$V_{eff} I_{eff} \cos(\theta - \phi)$	watt (W)
Reactive power	Q	$V_{eff} I_{eff} \sin(\theta - \phi)$	volt-ampere-reactive (VAR)
Complex power	$\mathbf{S}$	$P + jQ$	
		$V_{eff} I_{eff} \underline{/\theta - \phi}$	volt-ampere (VA)
		$\mathbf{V}_{eff}\mathbf{I}_{eff}^{*}$	
Apparent power	$\lvert\mathbf{S}\rvert$	$V_{eff} I_{eff}$	volt-ampere (VA)

The Power Triangle



$\theta - \phi > 0$: power factor is lagging (inductive load)

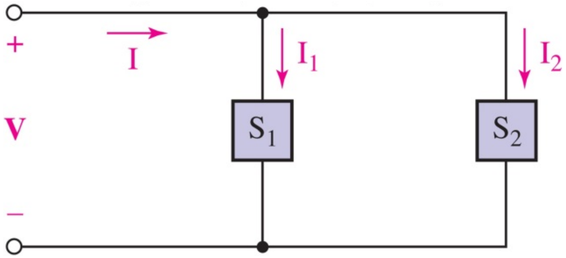
$\theta - \phi < 0$: power factor is leading (capacitive load)



- Real power (average power) : the magnitude of voltage phasor × the magnitude of current phasor which is in phase with the voltage.

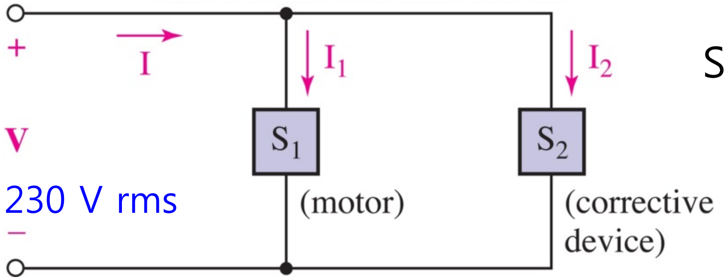
- Reactive power (quadrature power) : the magnitude of voltage phasor × the magnitude of current phasor which is 90° out of phase with the voltage.

Power Measurement



$$\boldsymbol{S} = \mathbf{V}\mathbf{I}^* = \mathbf{V}(\mathbf{I}_1 + \mathbf{I}_2)^* = \mathbf{V}(\mathbf{I}_1^* + \mathbf{I}_2^*) = \boldsymbol{S}_1 + \boldsymbol{S}_2$$

Example 11.9 An industrial consumer is operating a 50 kW induction motor at a lagging PF of 0.8. The source voltage is 230 V rms. In order to obtain lower electrical rates, the customer wishes to raise the PF to 0.95 lagging. Specify the suitable solution.



S_1 : 50 kW induction motor at a lagging PF of 0.8

$$\mathbf{S_1} = \mathbf{V}_{eff}\mathbf{I}_{eff}^* = V_{eff}I_{eff}\angle\,\theta - \phi = \frac{P}{PF}\angle\,\theta - \phi$$
$$= \frac{50k}{0.8}\angle\cos^{-1}0.8 = \frac{50k\angle 36.9°}{0.8} = 50 + j37.5\,kVA$$

In order to achieve PF of 0.95

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2} = \frac{50k}{0.95}\angle\cos^{-1}0.95 = 50 + j16.43\,kVA$$

$$\Rightarrow \mathbf{S_2} = \mathbf{S} - \mathbf{S_1} = 50 + j16.43 - 50 + j37.5 = -j21.07\,kVA$$

$$\Rightarrow \mathbf{S_2} = \mathbf{V}\mathbf{I}_2^* \Rightarrow \mathbf{I}_2^* = \frac{\mathbf{S_2}}{\mathbf{V}}$$
$$= \frac{-j21.07 \times 10^3}{230} = -j91.6\,A$$

$$\Rightarrow \mathbf{Z_2} = \frac{\mathbf{V}}{\mathbf{I_2}} = \frac{230}{j91.6} = -j2.51\,\Omega$$

TABLE 11.2 A Summary of AC Power Terms

Term	Symbol	Unit	Description
Instantaneous power	$p(t)$	W	$p(t) = v(t)i(t)$. It is the value of the power at a specific instant in time. It is *not* the product of the voltage and current phasors!
Average power	P	W	In the sinusoidal steady state, $P = \frac{1}{2} V_m I_m \cos(\theta - \phi)$, where θ is the angle of the voltage and ϕ is the angle of the current. Reactances do not contribute to P.
Effective or rms value	V_{rms} or I_{rms}	V or A	Defined, e.g., as $I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt}$; if $i(t)$ is sinusoidal, then $I_{\text{eff}} = I_m/\sqrt{2}$.
Apparent power	$\lvert\mathbf{S}\rvert$	VA	$\lvert\mathbf{S}\rvert = V_{\text{eff}} I_{\text{eff}}$, and is the maximum value the average power can be; $P = \lvert\mathbf{S}\rvert$ only for purely resistive loads.
Power factor	PF	None	Ratio of the average power to the apparent power. The PF is unity for a purely resistive load, and zero for a purely reactive load.
Reactive power	Q	VAR	A means of measuring the energy flow rate to and from reactive loads.
Complex power	$\mathbf{S}$	VA	A convenient complex quantity that contains both the average power P and the reactive power Q: $\mathbf{S} = P + jQ$.

Homework : 11장 Exercises 5의 배수 문제. 기말고사 칠 때 제출.

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