Lecture 10 – Market Structure (IV)

C. Game Theory

1. Foundations and Principles

1) Basic Elements of a Game

- *Players*: the identity of those playing the game, $N \geq 2$
- *Rules*: the timing of all players' move; the actions available to a player at each of her moves; the information that a player has at each move.
- *Outcomes*: It depends on what each player does when it is her turn to move. The set of outcomes is determined by all of the possible combinations of actions taken by players.
- *Payoffs*: It represents the players' preferences over the outcomes of the game.

2) Types of Games

- Static (strategic) games of complete information
- Dynamic games of complete information
- Static (strategic) games of incomplete information
- Dynamic games of incomplete information

3) Equilibrium Concepts

We want to focus on how to solve games. An equilibrium concept is a solution to a game. By this we mean that the equilibrium concept identifies, out of the set of all possible strategies, the strategies that players are actually likely to play. Solving for equilibrium is similar to making a prediction about how the game will be played. The focus is on defining commonly used equilibrium concepts and illustrating how to find strategies consistent with each concept.

4) Fundamental Assumptions

- *Rationality*: Players are interested in maximizing their payoffs.
- *Common Knowledge*: All players know the structure of the game and that their opponents are rational, that all players know that all players know the structure of the game and that their opponents are rational, and so on.

Static Games of Complete Information

"*Static*" means that players have a single move and that when a player moves, she does not know the action taken by her rivals. This may be because players move simultaneously. "*Complete information*" means that players know the payoffs of their opponents.

5) Normal Form Representation

- A set of players, identified by number: $\{1, 2, \cdots, I\}$
- A set of actions or strategies for each player i, denoted S_i. This is the "list" of permissible actions player i can take.
- A payoff function for each player i, $\pi_i(s)$, where $s = (s_1, s_2, \cdots, s_I)$ and $s_i \in S_i$ (strategy vector).
- In addition, the descriptions of some games require delineation of who knows what, when, and order of play, etc.

2. One Shot Game

If a game is played only once and the players move simultaneously or at least no player knows any of the other players' moves before choosing his. Thus we fully characterize a one-shot game by a list of the available strategies and payoffs $K = \{S_1, \cdots, S_I; \pi_1, \cdots, \pi_I\}$

1) Strategic Form

It is called the strategic (or normal) form representation of a game. For starters, let's consider the strategic form of a one-shot game with only two players, A and B, each with two strategies, 1 and 2. (The players could be two firms, an employer and employee, a parent and child, etc.) The payoffs for each player are collected in the following two matrices.

		Player B	
		1	***2***
Player A	***1***	π_A^{11}	π_A^{12}
	2	π_A^{21}	π_A^{22}

		Player B	
		1	***2***
Player A	***1***	π_B^{11}	π_B^{12}
	2	π_B^{21}	π_B^{22}

These are combined into a single *game matrix*:

		Player B	
		1	***2***
Player A	***1***	π_A^{11}, π_B^{11}	π_A^{12}, π_B^{12}
	2	π_A^{21}, π_B^{21}	π_A^{22}, π_B^{22}

,which fully summarizes the strategic form of the game. The game matrix is useful for depicting the strategic form of games with few players (usually two or three) and a finite number of strategies.

A game is *symmetric* if $\pi_A^{jk} = \pi_B^{kj}$ for all *j* and *k*. If $\pi_A^{jk} + \pi_B^{kj} = c$, where c is a constant, for each pair of strategies (*j*, *k*), then the game is *constant sum*; if *c = 0*, then it is a *zero-sum game*. Most generally, games are *variable sum*.

We are looking for a solution to such games. If each player is rational, what is her optimal strategy? This is given by the *best response function*. Player *i*'s best response to other player's strategies is the solution to the following maximization problem:

$$\max_{s_i} \pi_i(s_{1,}\cdots,\ s_{i-1},\ s_i,\ s_{i+1},\cdots,\ s_I) \qquad (i)$$

given the strategies of the $(I-1)$ other players. So the best response function is $s_i = R_i(\bar{s}_i)$, which can also be expressed as $R_i(s_1,\ \cdots,\ s_{i-1},\ s_{i+1},\ \cdots,\ s_I)$; that is, *i*'s best strategy is generally a function of the strategies of all other players.

If each player plays her optimal strategy, what happens? That is, what is the equilibrium of such a game?

2) Eliminating Dominated Strategies

One feature of best response functions is that they never reflect *dominated strategies*. For player *i*, strategy s_i' dominates strategy s_i'' if the payoff to s_i' exceeds the payoff to s_i'' for every combination of other players' strategies $\bar{s}_i$; that is, if

$$\pi_i(s_i', \bar{s}_i) > \pi_i(s_i'', \bar{s}_i) \qquad (ii)$$

for all $\bar{s}_i$. Rational players never play dominated strategies s_i'', so we can frequently eliminate some strategies as candidates for solutions.

3) **Pigs in a Box**

To see how this works, consider a game played by two pigs, one weak and the other strong. The pigs' strategies are simple: either Push a lever that distributes 100 calories of feed down a shoot about 20 feet away or Wait at the end of the feed shoot. Waiting burns no calories, but rushing from the lever to the shoot burns 10 calories. If both pigs wait, the lever is not pushed, so neither gets any feed. If the strong pig pushes the lever, he gets some of the feed by chasing the weak pig away; however, the weak pig cannot push the strong one aside, so the weak pig would get nothing by pushing the lever. Consequently, the caloric payoffs are given in the following game matrix:

		Strong Pig	
		Push	Wait
Weak Pig	Push	- 10, 90	- 10, 100
	Wait	75, 15	0, 0

What is the solution to this game? We begin by searching for dominated strategies to eliminate. Begin with the strong pig's best response: if the weak pig waits, the strong pig does better by pushing the lever; but if the weak pig pushes the lever, the strong pig's best response is to wait. So neither of the strong pig's strategies dominates the other. However, for the weak pig, waiting dominates pushing: 75 > - 10 and 0 > - 10. So we eliminate the weak pig's push strategy: it's dominated. Since only one strategy remains for the weak pig, that strategy is its *dominant strategy*. A strategy is dominant if it's the best response no matter what the other players' strategies are – that is, for every value of $\overline{s}_i$.

The solution to this game is a dominant strategy equilibrium. It is determined by eliminating all the dominated strategies. If what's left is unique, we have the equilibrium. Here waiting is the weak pig's dominant strategy. Given that the weak pig waits, the strong pig's best response is to push. Hence the equilibrium is the pair of strategies (Wait, Push) with associated payoffs (75, 15); that is, it's lower-left element of the game matrix. Only the weak survive!

4) **Prisoners' Dilemma**

		Clyde	
		Confess	Deny
Bonnie	Confess	- 3, - 3	0, - 10
	Deny	- 10, 0	- 1, - 1

This game is symmetric. Although the pair would be best off by both Denying guilt, Confess is a dominant strategy for each. Therefore, the dominant strategy equilibrium is the upper-left element of the game matrix. Even if the two were innocent!

5) **Cheating on the Cartel**

Suppose Coke and Pepsi consider to charge a high price for soda. If the game matrix were

		Pepsi	
		High *p*	Low *p*
Coke	High *p*	6, 6	2, 8
	Low *p*	8, 2	3, 3

with payoffs in millions of dollars of profit per week, each firm has a dominant strategy of cheating on the cartel by charging the low price. The dominant strategy equilibrium is the lower-right element: the cartel falls apart despite the joint advantage of charging the high price.

With larger strategy spaces, some strategies can be dominated without leaving a dominant strategy. In such cases, we eliminate dominated strategies iteratively. That is, some strategies that did not appear to be dominated do appear to be dominated once other dominated strategies are eliminated. That is true in the following example:

		Jack		
		Left	Middle	Right
Jill	Up	4, 1	3, 4	0, 1
	Down	1, 3	1, 2	2, 0

First, Jill does not have a dominant strategy. Her best response depends on whether Jack plays (Left or Middle) or Right. Second, but would jack ever play Right? No, Middle dominates Right (no matter what Jill does). So Right is a dominated strategy, and is irrelevant. Therefore, we have

		Jack	
		Left	Middle
Jill	Up	4, 1	3, 4
	Down	1, 3	1, 2

Third, having eliminated Jack playing Right, we now see that Jill playing Down can be eliminated: for Jill, the payoff to Up exceeds the payoff to Down no matter what Jack does. Since Jill's dominant strategy is Up, the dominant strategy equilibrium to this game is the upper-center element of the original game matrix.

To pull things together, our method for finding a solution to a one-shot game is first to eliminate dominated strategies and second to associate the solution with any dominant strategy that remains, if one exists.

6) **Nash Equilibrium**

Some games do not have a dominant strategy equilibrium. In these cases, we look for a strategically stable solution – one that none of the players would choose to deviate from. Strategies $s^* = (s_1^*, \cdots, s_I^*)$ are a Nash equilibrium if each strategy s_i^* is a best response to the other strategies. That is, s_i^* solves

$$\max_{s_i} \pi_i(s_1^*, \cdots, s_{i-1}^*, s_i^*, s_{i+1}^*, \cdots, s_I^*) \qquad (iii)$$

To start simply, let's consider the following 2×2 game:

		Jack	
		Left	Right
Jill	Up	3, 2	1, 1
	Down	1, 1	2, 3

First, there are no dominant strategies. Second, there is a Nash equilibrium. To see this, suppose Jill plays Up, so Jack's best response is Left and Jill's best response is Up. Each is a best response to the other, so (Up, Left) is a Nash equilibrium. One can show that (Down, Right) is also a Nash equilibrium. However, the other two pairs are not.

(Please review Cournot and Bertrand Duopoly models again! Lec 08. pp. 34-38)

7) Battle of the Sexes

In Jack and Jill’s 2×2 game, there are multiple Nash equilibria. Likewise, the following Battle of the Sexes game has two Nash equilibria.

		Ethel	
		Boxing	Opera
Fred	Boxing	4, 3	1, 1
	Opera	0, 0	6, 6

Fred and Ethel choose strategies simultaneously without coordination. Clearly, they like to be together: going separate ways is not a Nash equilibrium. But there are two Nash equilibria: both go to the boxing match or both go to the opera. Unlike Jack and Jill’s game, one Nash equilibrium dominates the other. So as a refinement of Nash equilibrium, we expect Fred and Ethel to go to the Opera. If multiple Nash equilibria cannot be ranked, then there’s no telling what might happen.

8) Mixed Strategies

Must there be at least one Nash equilibrium? Yes, but we might not find it in **pure strategies**, which is what we’ve analyzed so far. More generally, players can randomize their strategies. These are called **mixed strategies**. Suppose we find no Nash equilibrium. We then need to check best responses that take the form of a probability associated with each strategy. This is best understood in the context of sports competition.

Consider a simplified version of the game played by a pitcher and a batter in baseball. The pitcher has two strategies regarding the pitch he will throw: Fastball or Curve. The batter’s two strategies are: Rush or Wait. If the batter guesses wrong, he’s sure to make an out; if he correctly guesses curve ball, he’s sure to get a hit; if he correctly guesses fastball, however, his probability of getting a hit is p. So the game matrix is:

		Batter	
		Rush	Wait
Pitcher	Fastball	$1-p$, p	1, 0
	Curve	1, 0	0, 1

for this zero-sum game. One can verify that there is no Nash equilibrium in pure strategies. For instance, if the pitcher chose to throw a curve, the batter would wait, which implies that the pitcher wouldn’t want to throw a curve.

To find the Nash equilibrium in mixed strategies, we also assume that each player is an expected utility-maximizer. If the pitcher throws a fastball, his expected payoff would be $\pi_R(1-p)+(1-\pi_R)\cdot 1$, where π_R is the probability the batter rushes. Similarly, if the pitcher throws a curve, his expected payoff would be π_R. Therefore, the pitcher’s best response function is the solution to:

$$\max_{\pi_f} \pi_f[\pi_r(1-p)+(1-\pi_r)]+(1-\pi_f)\pi_r \qquad (iv)$$

given π_r. The solution to this problem is

$$\pi_r^* = \frac{1}{1+p} \qquad (v)$$

That is, to make the pitcher indifferent between throwing fastballs and curve balls, the batter’s probability of rushing must be π_r^*.

We must also solve the batter’s problem. The batter chooses π_r to maximize his expected utility given the pitcher's mixed strategy. The solution is:

$$\pi_f^* = \frac{1}{1+p} \qquad (vi)$$

If p = ½, fastballs would be thrown and anticipated two-third of the time. Notice, the better the batter is at hitting an anticipated fastball, the less likely he'll see one. What's the batter's expected batting average? $\left(\frac{p}{1+p}\right)$. Do better batters have higher equilibrium batting averages?

(Yes, but the pitcher's response attenuates the effect of skill on batting average.)

9) **Remarks**

Several results are worth remembering. First, all dominant strategy equilibria are Nash equilibria. Second, not all Nash equilibria are dominant strategy equilibria. Third, there always exists at least one Nash equilibrium, although perhaps not in pure strategies. Fourth, if there's a single Nash equilibrium, that's our prediction. Fifth, if there exist multiple Nash equilibria, eliminate Pareto-dominated Nash equilibria.

3. Repeated Games

Suppose a one-shot game G is repeated T times. Let $G(T)$ denote this repeated game. This expands the strategy space for each player. Player i's strategy would be a sequence of moves, one move for each round. More precisely, a player's strategy specifies the action to be taken in each stage for each possible history of play through the previous round. This introduces the possibility of reputations, threats, and rewards.

The method for analyzing repeated games, as well as sequential games, is backward induction – look forward and reason backward.

1) Fixed Repetitions

If T is fixed, how would the solution to $G(T)$ be related to the solution to G? Would the equilibrium change? Suppose $T = 2$. Clearly, the solution in the second round must be the solution to the one-shot game. Now step back to the first round. Each player looks forward to see that the solution to the second round will be the solution to the one-shot game. That is, payoffs in the second round will not depend on play in the first round. Therefore, best responses in the first round are the one-shot game best responses. So, the game unravels.

Result: The solution to a repeated game with fixed repetitions is the sequence of solutions to the one-shot game, if the solution to the one-shot game is unique.

This is sometimes called the *end-game problem*.
If the solution to the one-shot game is not unique, play in the first round can influence which of the multiple equilibria is played in the second round.

2) Indefinite Repetitions

If the horizon were infinite – or the game were repeated a random number of times – the repeated game would not unravel, and richer strategies might comprise the equilibrium. Consider a trigger strategy: if I play Nice until you play Nasty, then I play Nasty forever. Both players following a trigger strategy is a Nash equilibrium – and each would play Nice in each round – but would it really be in my interest to follow through on my threat if you do play Nasty? It turns out that this isn't a problem. The Folk Theorem (Friedman 1971) guarantees that a trigger strategy works in supporting a large number of outcomes as long as the players don't discount the future too heavily.

Alternatively, I might play Tit-for-Tat: penalize your Nasty-playing opponent for only one round; that is, your strategy in a round is your opponent's strategy in the previous round. Experimental evidence suggests that tit-for-tat is hard to beat. But it's an awful strategy if there's a possibility of mistakenly accusing the other player of playing Nasty.

3) Cartel Enforcement

Enforcing a collusive agreement is more likely to be successful in long-term relationships. In the one-shot game, as we saw above, cheating is the dominant strategy. If the cartel has a known horizon, again every member cheats. Since penalties in an infinitely repeated game can support the cartel equilibrium, cheating can be deterred. However, the members must not discount the future too heavily; in particular, the probability that $G(T)$ ends can't be too high. And, of course, precise detection is assumed.

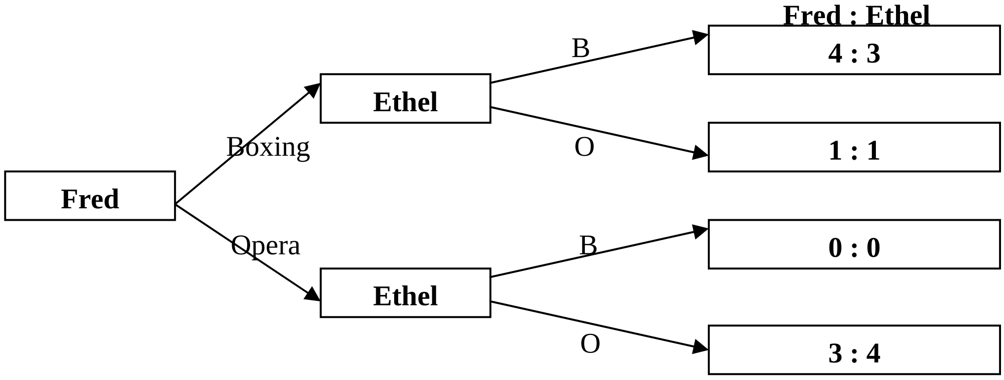
4. Sequential Games

The games we've studied to this point have been simultaneous move games, or at least no player knows another's move before making his own. Now we turn to games where at least some of the moves are taken in turn.

1) Extensive Form

A game's extensive form specifies the players, when each player has a move, what each player can do when it's his move, what each player knows when it's his move, and each player's payoffs for each combination of moves of all players. *The key difference between strategic and extensive form representations of a game is that the latter specifies the order of play.* This is essential in sequential games.

The game tree depicts a game's extensive form. For instance, a game tree for the Battle of the Sexes is



The game tree begins with a *decision node* for one player and ends with *terminal nodes*, which list the payoffs associated with a series of strategies. In between are decision nodes for one player or another. Along the branches, we specify the strategies.

If Fred and Ethel play simultaneously, we add the shaded information box to indicate that Ethel does not have knowledge of Fred's strategy before she plays. (We could have put Fred after Ethel, too.) In this case, the only extra information in the game tree is that the moves are indeed simultaneous.

Alternatively, if Fred moves first, we remove the information box. Here the game tree specifies the order of play, but the game matrix does not: the strategic form representation suppresses the order of play. To the extent order of play is important in determining the equilibrium, we must use the extensive form.

As in the one-shot version of this game, there are two Nash equiliria. Order of play has no effect on the Nash equilibria. However, not all Nash equilibria are sensible in sequential games. To see this, let's solve the game by backward induction. We start by solving each of Ethel's two *subgames*. If Fred chose Boxing, the solution to the subsequent subgame is (Boxing, Boxing). If Fred chose Opera, the solution to the subsequent subgame would be (Opera, Opera). Since Ethel will condition her choice on Fred's and Fred knows this, his best response is to choose Boxing. That is, he knows that Ethel will follow him, so he chooses his favorite.

2) Subgame Perfection

This game is *subgame perfect*. To be subgame perfect, the players' strategies must constitute a Nash equilibrium in every subgame – that is, forward from each decision node. Although going together to the Opera is a Nash equilibrium, it is not subgame perfect. The idea here is that subgame perfection throws out Nash equilibria that could be supported by only incredible threats. For instance, Ethel could announce that her strategy will be Opera. If Fred believed her, he would also choose Opera. But Ethel's strategy isn't *credible*. If Fred chose Boxing, it wouldn't be her best response to choose Opera. So choosing Opera in that subgame wouldn't be a Nash equilibrium, which implies going together to the Opera is not subgame perfect.

(Review the Stackelberg Duopoly for your better understanding! Lec 08. pp.39-41)

3) Commitment

If Ethel could commit to go to the Opera, she would win the Battle of the Sexes. To commit, she must take it her best response to go to the Opera no matter where Fred goes. That is, she must reduce her options to improve her payoffs.

Suppose she posts a bond of $3 with Lucy. If she goes to the Boxing match, she forfeits the bond. This changes Ethel's payoffs. Now Opera emerges as a dominant strategy in the second subgame. Her threat is credible. Fred's best response ex ante is to go to the Opera. Ethel wins. Adding the possibility of commitment would expand the strategy space (e.g., how much to pay Lucy), so Fred and Ethel would be playing a different game.