Lecture 11 – Uncertainty

1. Contingent Claims and the State-Preference Model

1) Contingent Commodities and Contingent Claims

Using the simple two-good model we have developed throughout this course, think of x_1 as consumption when state of the world 1 occurs and x_2 as consumption when state of world 2 occurs. We call a commodity that is to be delivered in only one state of the world a **state-contingent commodity** or just **contingent commodity**. Actually, if there are *n* different commodities and *m* states of the world, there will be $(n \times m)$ different contingent commodities. For now, we consider only one commodity (consumption) and two states of the world: "good" (1) and "bad" (2). Thus, in this case, there are two contingent commodities.

In the state-preference model, consumers trade contingent claims, which are rights to consumption, if and only if, a particular state of the world occurs. Thus, in this framework, x_1 represents the amount of the good the consumer will receive (or purchase) if state 1 occurs and x_2 is the amount s/he will receive if state 2 occurs.

Ex) Betting in a horse race. The states of the world correspond to how the various horses will place, and a claim corresponds to a bet that a horse will win. If your horse comes in, you get paid in proportion to the number of tickets you purchased. But the only way to guarantee payment in all states of the world is to bet on all the horses.

*2) Risk Sharing between Consumers in a Contingent-Claims Market

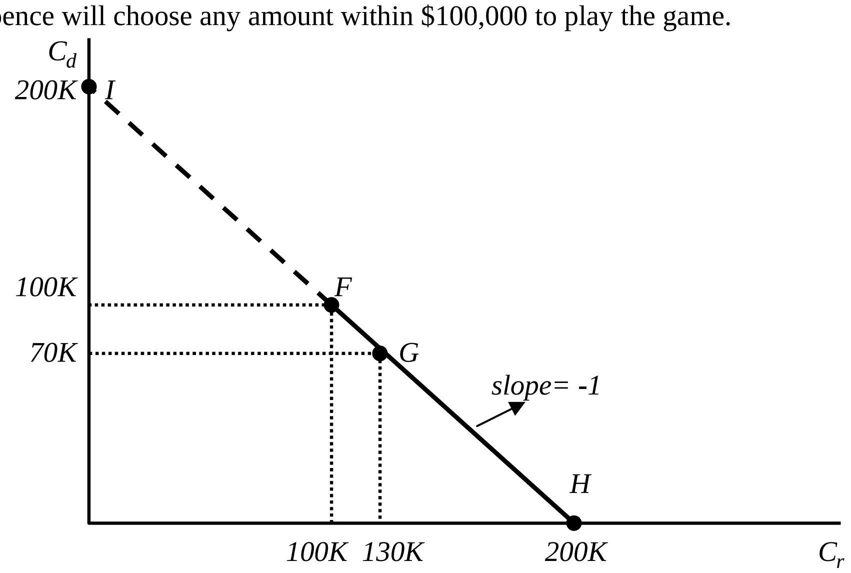
For example, Frank Knight is a dealer in a Casino. Michael Spence wants to play the card game with $100,000.

There are 3 hidden cards, two of which have dandelions and the remaining one has a rose. If Spence picks a card with rose, he will win the amount that he bet. And if Spence picks a card with dandelion, he will lose the amount that he bet.

Let C_r and C_d denote the two possible commodity bundles that Spence will shop after the card game (contingent commodity).

• Budget Line

Spence will choose any amount within $100,000 to play the game.



F: initial endowment, $\overline{FH}$: Budget Set (or opportunity set)
How about $\overline{IF}$?

• Iso-Expected Value Line

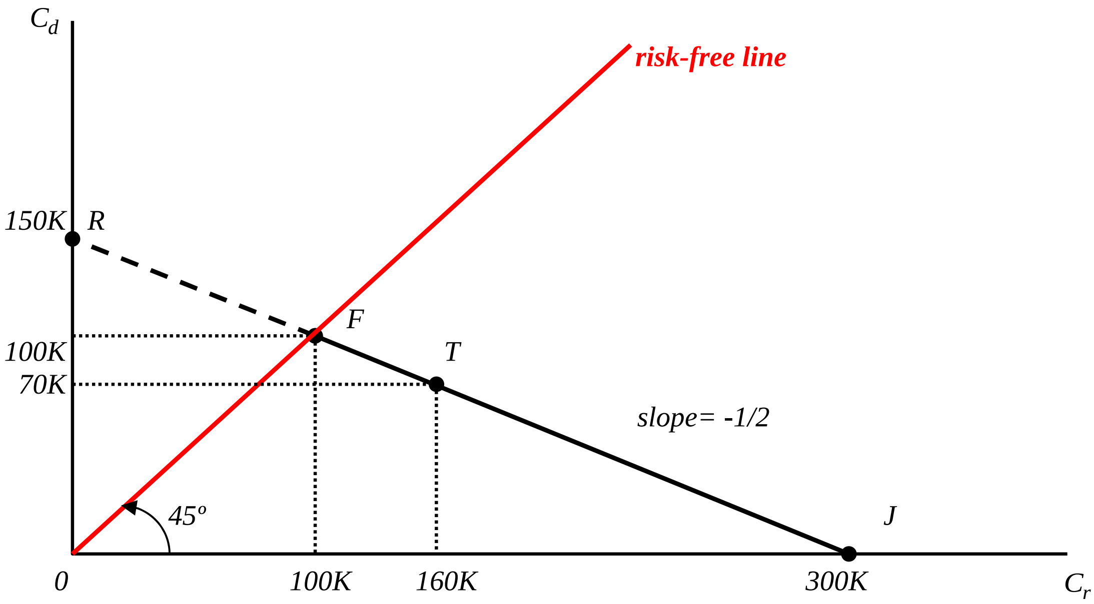
Now, based upon the game framework, we can calculate the *expected value* of the consumption that Spence will take.
Since the probabilities are 1/3 and 2/3 for rose and dandelion, respectively, if Spence bet \$ 30K, then $C_r = 130K$ and $C_d = 70K$. So the expected value if he bet \$ 30K will be:

$$E = \left(\frac{1}{3} \times 130{,}000\right) + \left(\frac{2}{3} \times 70{,}000\right) = 90{,}000$$

Is this card game fair? Absolutely not. Let's compute the expected value if Spence bet all the money he had (\$100,000). How much can he expect to make?

$$\left(\frac{1}{3} \times 200{,}000\right) + \left(\frac{2}{3} \times 0\right) = 66{,}667$$

Now, how can we design a new rule that can make this game *fair* (a game that can guarantee the initial state regardless of how much money player bets. Net gain is always zero)?
Simply because there is one card of rose along with two cards of dandelions, the way to make a fair game is to double the award when player picks a card of rose (we got a new game rule!)
Can you draw a new budget line with the new framework? Sure, you can.



$\overline{FJ}$: *iso-expected value line* or *fair odds line*

• Meaning of Risk

It is risk free not to play the game (45° line is called risk-free line).
The more money the players bet, the higher the risk s/he needs to take.
Moving farther from F, which is risk free, the riskiness will rise.
Point J represents the maximized level of riskiness with given condition.

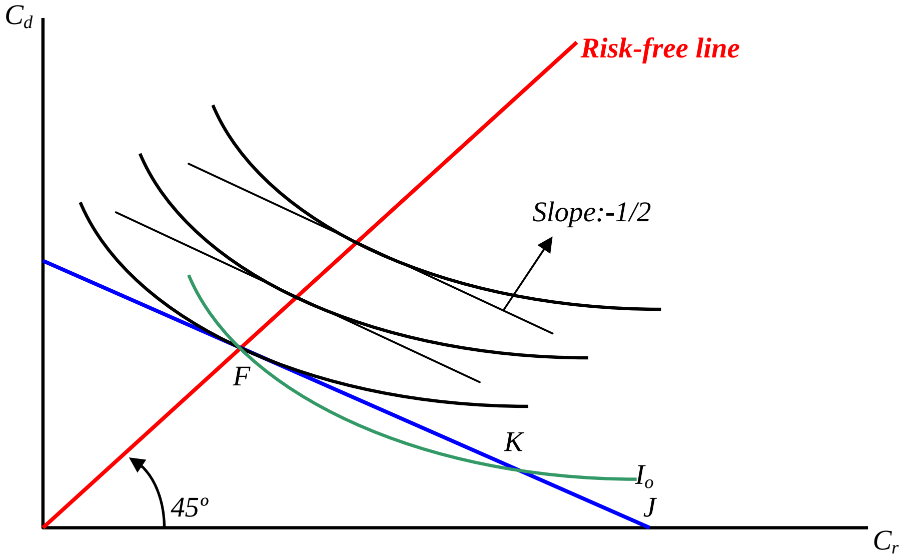
• Preference and Choices of Decision-makers

Risk neutral person will rely only upon the expected value of the game. If the game is

favorable based on the computed expected value, s/he will play.
Risk averse person will not play unless the game is sufficiently favorable. So s/he will not play even if the game is fair with 0 expected value.
Risk loving person will play even though the game is disadvantageous based on the expected value.

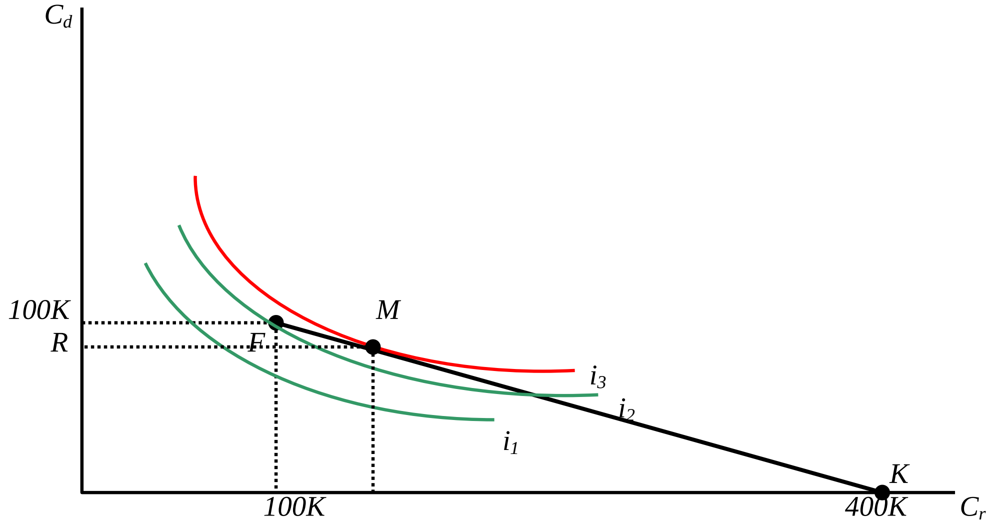
• Indifference Curves with Risk Averseness

Convex indifference curve implies that people prefer the average consumption bundles to the extreme ones. And the slope of indifference curve on 45º line (risk-free line) will have a same slope with the fair odds line. Why? (explain with the green indifference curve I_o)



So, we can presume that the person with risk averseness will choose the point F as the optimal point if the game is fair (will play the game).

Now, what if the game is advantageous to the player?

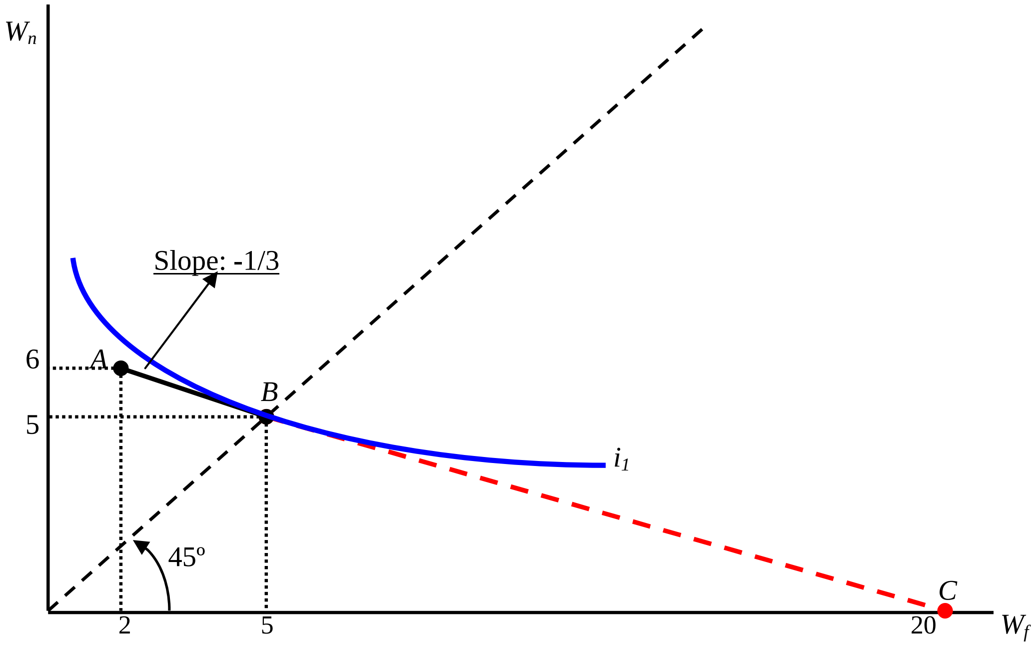


Even if Frank offers three times of the amount that Michael bet, Michael will bet small amount of money (*100K – R*).

***3) Application: Insurance Market**

• Model

Total initial assets: $6 million
Probability of fire: 25% (or ¼)
Estimated losses due to fire: $4 million. So the new assets after fire will be $2 millions.
W_f (fire) and W_n (no fire) are the contingent commodities.
Reimbursement in case of fire: $4 per $1 of insurance premium.



In reality, however, it is not reasonable that reimbursement is bigger than the actual losses. So, it makes sense that if insured pays $1 million to be reimbursed with $ 4 millions, which will make even between the case with and without fire damages (as long as s/he is insured). Therefore, the budget line (and fair odds line, as well) that this person is facing would be $\overline{AB}$ ($\overline{BC}$ is not feasible at all).

Generally, if the ***premium rate*** (ratio of premium to reimbursement) equals the probability of damage (fire, losses, etc), that insurance program is called "***fair insurance***." In this case, the insurance premium amounts will equal the expected value of reimbursement amounts.

In our example, the probability of fire was assumed to be 25% (or ¼). So this insurance will be fair. And the absolute value of the slope of $\overline{AB}$ is 1/3, which implies that this insurance is fair. This slope means the ***fair odds*** under the condition that probability of fire is ¼ and probability of safety is ¾. Based upon this fair insurance, the expected value of assets of the insured on $\overline{AB}$ will be always $5 millions.

Since this insured is risk averse person (because s/he wants to buy insurance), his or her best choice must be on point *B*.

2. Theory of Expected Utility

1) St. Petersburg Paradox

Suppose you had the opportunity to pay $100 and then play of the following gambles, each of which is fair bet.

- You get back $100.
- I toss a fair coin. You receive $200 if heads.
  Or 0 if tails.
- I roll a fair die. You receive
  $400 if 1, $70 if 2, $55 if 3, $25 if 4, $40 if 5, and $10 if 6.

All the gambles described above have expected value of $100, but would you be equally willing to play each one? For one thing, the variances are different:

- 0
- $\frac{1}{2}(200-100)^2+\frac{1}{2}(0-100)^2=10{,}000$
- $\frac{1}{6}\left(300^2+30^2+45^2+75^2+60^2+90^2\right)=18{,}375$

You might be more willing to play the gamble with the lower variance than the one with the higher variance.

This point is illustrated by what is called the St. Petersburg paradox. This was noted by Bernoulli, a Swiss mathematician of the 18th century. He proposed a variation of the following gamble. Suppose a fair coin is tossed until it comes up heads. Your payoff depends on the number of tosses before heads appears for the first time. Recognizing that tosses of a fair coin are independent and that probabilities get multiplied together on successive tosses, your payoffs in Bernoulli's game are constructed as follows:

$2 if heads comes up first on the first try ($p=1/2$)
$4 if heads comes up first on the second try ($p=1/4$)
$8 if heads comes up first on the third try ($p=1/8$)
⋮
$ 2^n if heads comes up first on the *n*-th try ($p=1/(2)^n$)
⋮

The expected value of the gamble set out above is

$$\left(\frac{1}{2}\right)2+\left(\frac{1}{4}\right)4+\left(\frac{1}{8}\right)8+\cdots+\left(\frac{1}{2^n}\right)2^n+\cdots=\sum_{n=1}^{\infty}\frac{1}{2^n}2^n=1+1+1+\cdots=\infty$$

But no one would pay an infinite amount to play this gamble. In fact, few would play much more than a few dollars. One reason might be that the variance of this gamble is also infinite, and *most people prefer lower variance (less uncertainty) to more.*

*2) Expected Utility and the von Neumann-Morgenstern Utility Function

The St. Petersburg paradox suggests that we need some concept other than expected value to analyze how people make decisions in risky situations. In 1944, John von Neumann and Oscar Morgenstern used such a concept, called expected utility, in their book on game theory (*Theory of Games and Economic Behavior*). **Expected utility** is a representation of preference under uncertainty in terms of the expected value on a set of utilities over possible outcomes, x_i :

$$E\{U\} = \sum_{i=1}^{n} p_i U(x_i) \quad (i)$$

,which is linear in the probabilities (p_i).

3) The Axioms of Expected Utility

- Preferences over possible outcomes are *complete, reflexive,* and *transitive*.
- *Compound lotteries can be reduced to simple lotteries.*
- *Continuity*. For each outcome x_i between x_1 and x_n, the consumer can name a probability, p_i, such that s/he is indifferent between getting x_i with certainty and playing a lottery (which involves getting x_n with probability p_i and x_1 with probability $(1-p_i)$). We say that x_i is the certainty equivalent to the lottery, $\tilde{x}_i$, where

$$\tilde{x}_i = (x_n \text{ with } p_i \text{ and } x_1 \text{ with } (1-p_i)).$$

- *Substitutability*. The lottery $\tilde{x}_i$ can always be substituted for its certainty equivalent x_i in any other lottery.
- *Monotonicity*. If two lotteries with the same two alternatives each differ only in probabilities, then the lottery that gives higher probability to the most-preferred alternative is preferred to the other lottery.

If preferences over lotteries satisfy above axioms, then we can assign numbers $U(x_i)$ which is associated with the outcomes x_i, such that if we compare two lotteries L and L' which offer probabilities $(p_1, \cdots, p_n)$ and $(p_1', \cdots, p_n')$ of obtaining those outcomes, L will be preferred to L' if and only if

$$\sum_{i=1}^{n} p_i U(x_i) > \sum_{i=1}^{n} p_i' U(x_i)$$

This means that the rank order by expected utilities reflects the rank order of preference over the lotteries and that the rational individual will choose among risky alternatives as if s/he is maximizing expected utility.

*4) The von Neumann-Morgenstern Utility Function

A lottery L will pay \$A with probability p and \$B with $(1-p)$. If we can assign numbers (cardinal utilities) $U(A)$ and $U(B)$, then the expected utility of this lottery L can be defined as follows:

$$U(L) = pU(A) + (1-p)U(B)$$

By assigning two (hypothetical) extreme numbers A, B, $U(A)$ and $U(B)$, respectively.
Suppose that $A = \$100{,}000$ and $B = -\$3{,}000$. Also $U(A) = 100$ and $U(B) = 0$
Using the *continuity* axiom, we can calculate the specific probability p that can make this decision-maker indifferent between buying this lottery (risky asset) and having any specific amount of money (safe asset). Let's take \$25,000 as safe asset. If this person says that his p equals 0.4, then the utility that \$25,000 will create is

$$U(\$25{,}000) = 0.4 \times U(\$100{,}000) + (1-0.4) \times U(-3{,}000) = 0.4 \times 100 + 0.6 \times 0 = 40$$

Using this example, the expected value of the award from this lottery is calculated simply $\$[pA + (1-p)B]$. The risk averse person will prefer the situation of having the expected value amount for sure to playing this lottery. In his viewpoint, the expected utility of lottery is smaller than the utility of expected value.

$$pU(A) + (1-p)U(B) < U[pA + (1-p)B]$$

This characteristic is analogous to the mathematical implication of "*strictly concave function,*" which means that risk averse person will have (strictly) concave utility curve.

• Concave Function

Suppose there is a function $y = f(x)$. Take any two values of x_1, x_2. And let x_3 be located in between x_1 and x_2. x_3 can be mathematically expressed as follows:

$$x_3 = kx_1 + (1-k)x_2,\ 0 < k < 1$$

Hence, x_3 is a *convex combination* of x_1 and x_2. If k is close to 0, x_3 approaches to x_2.

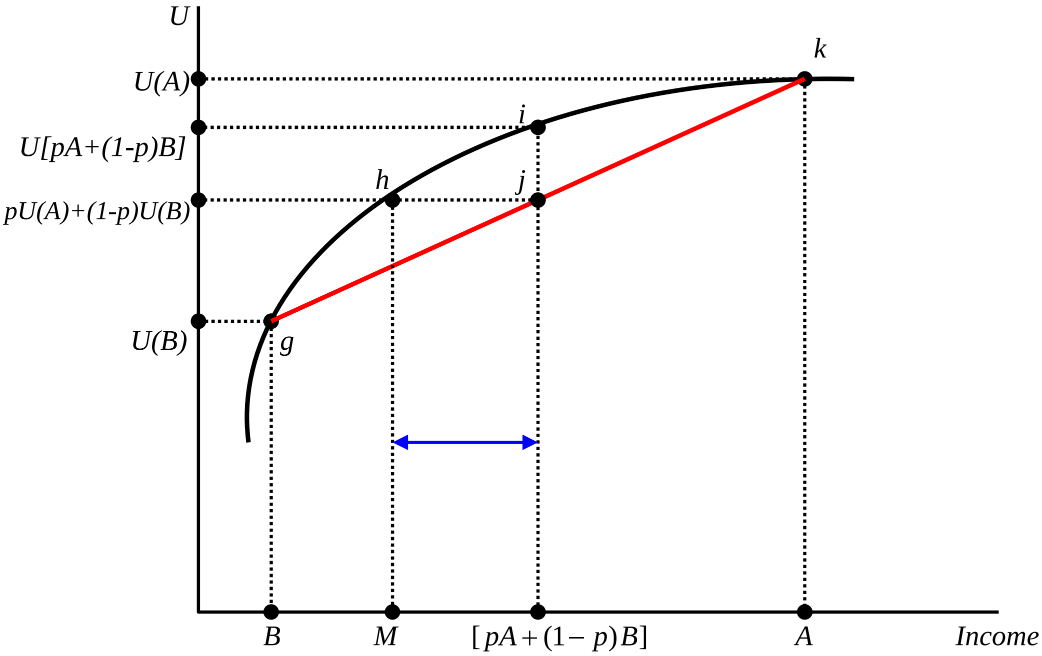
Concave function can be expressed as

$$kf(x_1) + (1-k)f(x_2) < f[kx_1 + (1-k)x_2]$$

Likewise, convex function can be expressed as

$$kf(x_1) + (1-k)f(x_2) > f[kx_1 + (1-k)x_2]$$

• Determining risk premium



5) Portfolio

All types of assets that people diversify. So, the important issue in portfolio is the portion of each type of asset. How to choose (or allocate) the combination of riskiness and rate of return will depend upon each person's attitude toward riskiness or uncertainty.

• Portfolio Selection Model

Two types of assets: safe vs. risky assets

r_f : rate of return of safe (or risk-free) asset

r_k : rate of return of risky asset.

We can simply imagine that $r_f < r_k$.

To express the level of riskiness, we can adopt the *standard deviation* of the rate of return (σ).

So, let's assume that r_k has the standard deviation of σ_k.
Suppose a person has $1 worth of wealth and s/he wants to have a as a type of risky asset and $(1-a)$ as safe asset. Based on this information, we can calculate the rate of return of this portfolio (r_p) and level of riskiness (σ_p) as follows:

$$\begin{cases} r_p = ar_k + (1-a)r_f & (ii) \\ \sigma_p = a\sigma_k & (iii) \end{cases}$$

Finally, the question is to get the specific value of a in this portfolio selection model.
Using utility function with the insertion of two factors, r_p and σ_p, we can set up

$$U = U(r_p,\ \sigma_p) \qquad (iv)$$

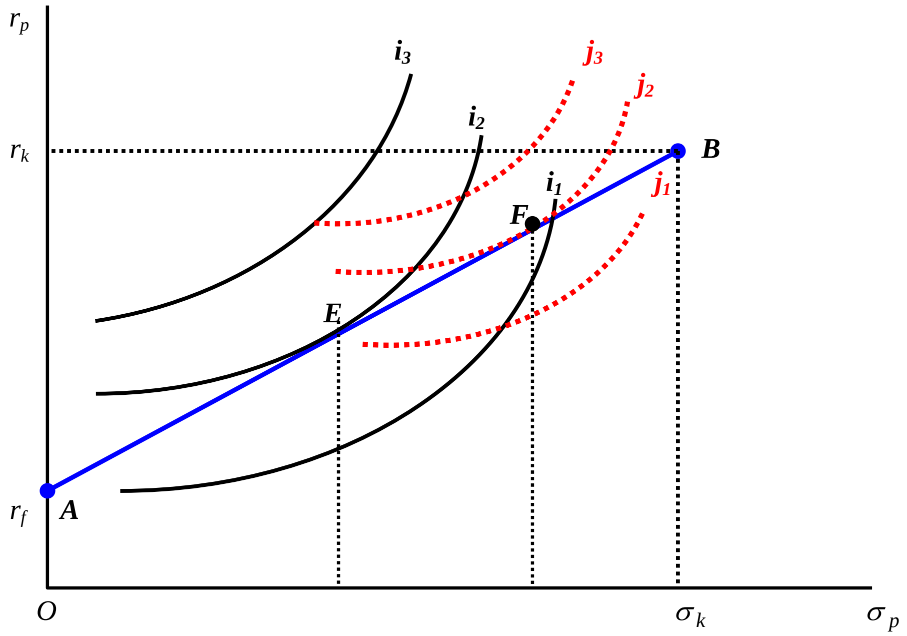
If this person is risk averse, s/he thinks that σ_p will be a "bad."

From (*ii*), we can get

$$r_p = r_f + a(r_k - r_f) \qquad (v)$$

Plugging $a = \sigma_p / \sigma_k$ into (*v*), we can get the final expression

$$r_p = r_f + \frac{r_k - r_f}{\sigma_k} \cdot \sigma_p \qquad (vi)$$



A: $a = 0$. B: $a = 1$

$\frac{r_k - r_f}{\sigma_k}$: ***price of risk***