5.3.6 Time Reversal

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n]e^{-j\omega n}$$

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega})$$

$$x[-n] \overset{\mathfrak{I}}{\longleftrightarrow} X(e^{-j\omega})$$

5.3.7 Time Expansion

For Continuous-time signals

$$x(at) \overset{\mathfrak{I}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

It is different for discrete-time signal.

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x_{(k)}[n] \overset{\Im}{\longleftrightarrow} X(e^{jk\omega})$$

Proof)

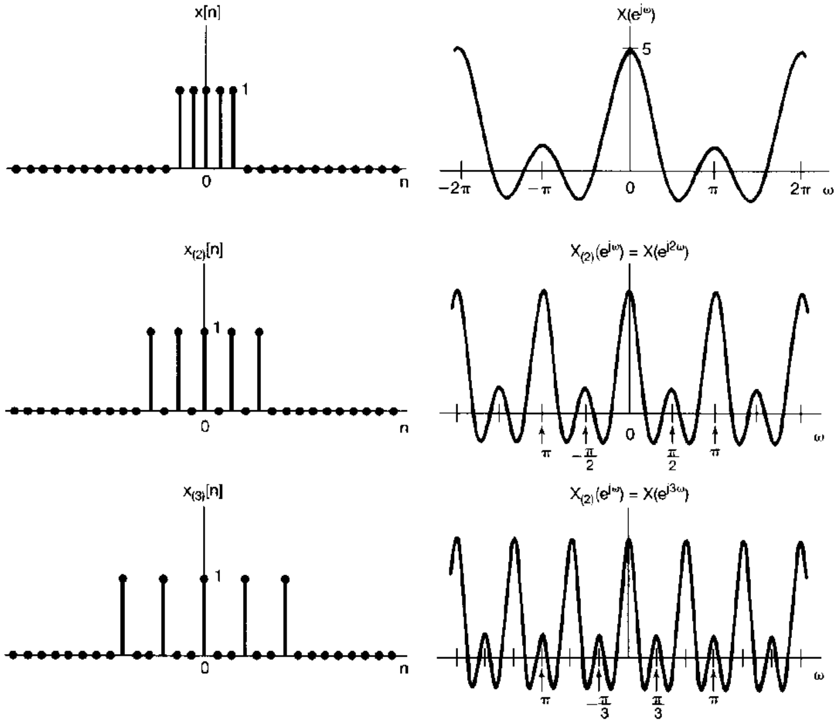
$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n}$$

$$n = \begin{cases} rk \\ rk+1, rk+2, \cdots, rk+(k-1) \end{cases}$$

$$= \sum_{r=-\infty}^{\infty} x_{(k)}[rk] e^{-j\omega rk}$$

$$= \sum_{r=-\infty}^{\infty} x[r] e^{-j\omega rk}$$

$$= \sum_{r=-\infty}^{\infty} x[r] e^{-j(k\omega)r}$$



5.3.8 Differentiation in frequency

$$nx[n] \overset{\mathfrak{I}}{\longleftrightarrow} j\frac{\mathrm{d}X(e^{j\omega})}{\mathrm{d}\omega}$$

5.3.9 Parseval's relation

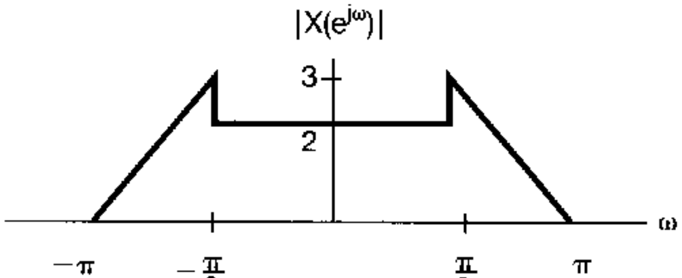
$$\sum_{n=-\infty}^{+\infty} \left| x[n] \right|^2 = \frac{1}{2\pi}\int_{2\pi} \left| X(e^{j\omega}) \right|^2 d\omega$$

$\left| X(e^{j\omega}) \right|^2$; the energy density spectrun of $x[n]$

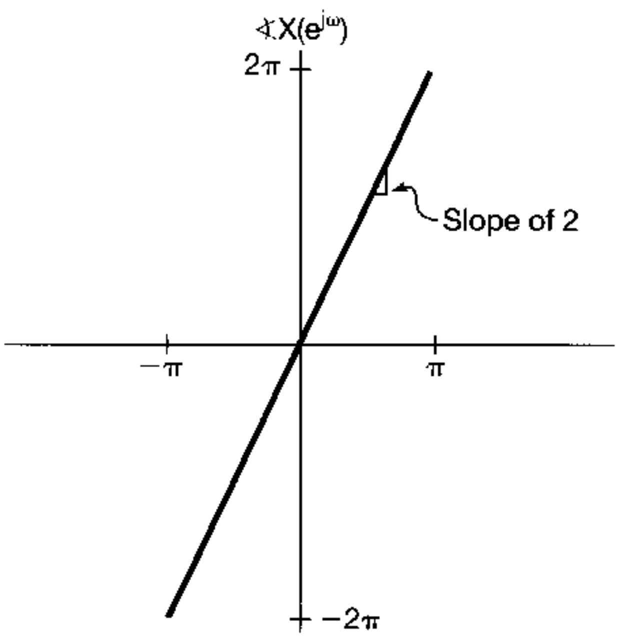
Ex. 5.10)

Is the time-domain signal $x[n]$

Periodic?
Real?
Even?
Of finite energy?



(a)



(b)

5.4 The Convolution Property

$$y[n] = x[n] * h[n] \quad \Longleftrightarrow \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Example : convolution of two sequences

$$x[n] = \{3, 2, 1\}, \quad h[n] = \{1, 2, 1\} \qquad y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

$$X(e^{j\omega}) = 3 + 2e^{-j\omega} + e^{-j2\omega}$$

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

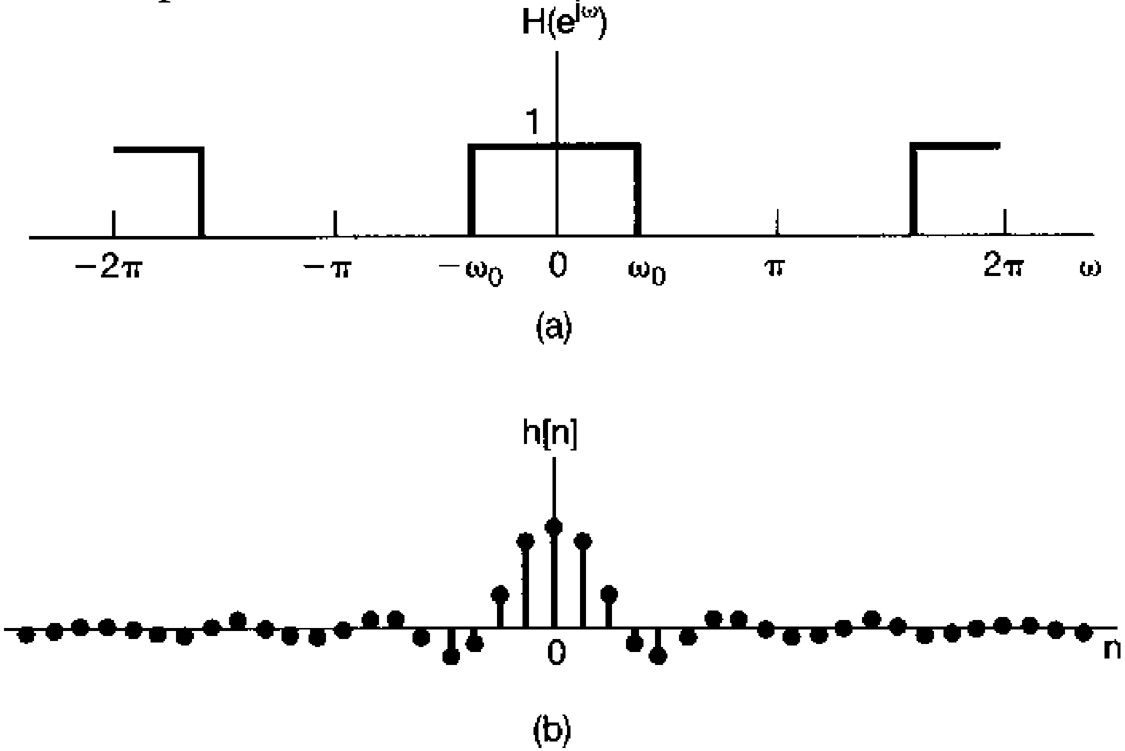
$$= 3 + (6+2)e^{-j\omega} + (3+4+1)e^{-j2\omega} + (2+2)e^{-j3\omega} + e^{-j4\omega}$$

Ex. 5.11)

$$h[n] = \delta[n - n_0]$$

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Ex. 5.12) Ideal lowpass filter



$$h[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}e^{j\omega n}d\omega=\frac{\sin\omega_0 n}{\pi n}$$

Ex. 5.13)

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} \qquad X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

$$Y(e^{j\omega}) = \frac{A}{(1-\alpha e^{-j\omega})} + \frac{B}{(1-\beta e^{-j\omega})}$$

$$y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n]$$
$$= \frac{1}{\alpha-\beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]$$

For $\alpha = \beta$, $\quad Y(e^{j\omega}) = \left(\dfrac{1}{1-\alpha e^{-j\omega}} \right)^2$

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}} \right)$$

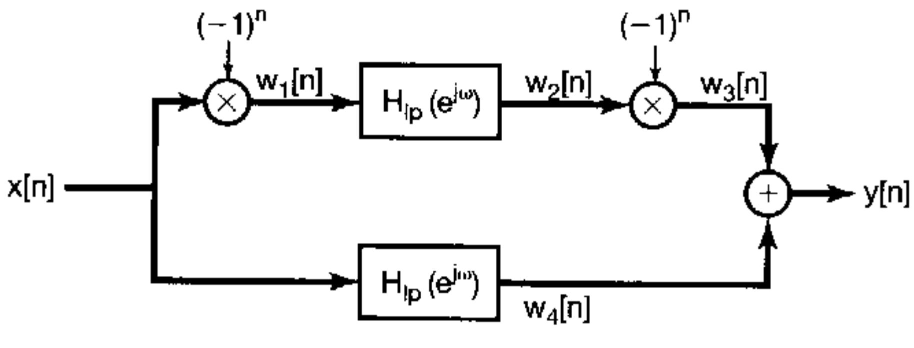
$$\alpha^n u[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} \left(\frac{1}{1-\alpha\ e^{-j\omega}} \right)$$

$$n\alpha^n u[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha\ e^{-j\omega}} \right) \quad \text{(Differentiation in frequency)}$$

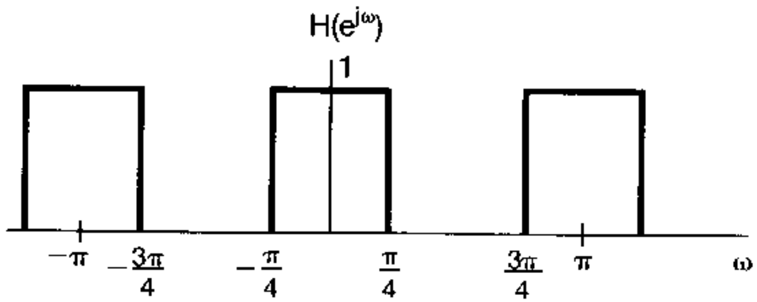
$$(n+1)\alpha^{n+1} u[n+1] \stackrel{\mathfrak{F}}{\longleftrightarrow} je^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha\ e^{-j\omega}} \right)$$

$$\therefore\ y[n] = (n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

Ex. 5.14) Ideal bandstop filter at $\frac{\pi}{4} < |\omega| < \frac{3\pi}{4}$



(a)



(b)

$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega})$$

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = \left[H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})\right]X(e^{j\omega})$$

$$\therefore\ H(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})$$

5.5 The multiplication property

$$y[n] = x_1[n]x_2[n]$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}$$

$$= \frac{1}{2\pi}\left\{\int_{2\pi} X_1(e^{j\theta})\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}d\theta\right\}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\left\{\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta\right\}$$

Ex. 5.15)

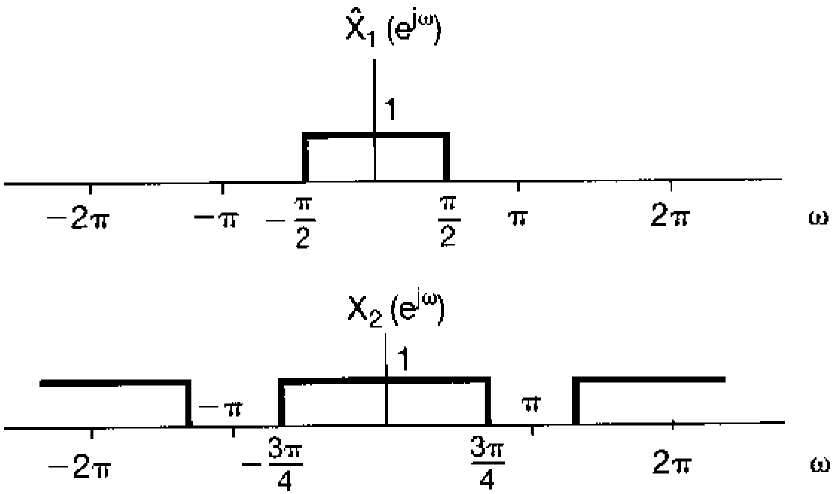
$$x[n] = x_1[n]x_2[n], \quad x_1[n] = \frac{\sin(\pi n/2)}{\pi n}, \quad x_2[n] = \frac{\sin(3\pi n/4)}{\pi n}$$ (Text Correction)

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & for \quad -\pi < \omega \le \pi \\ 0 & otherwise \end{cases}$$

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$



Result: Fig. 5.20, page 390

5.7 Duality

5.7.1 In the D.T. Fourier Series

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad\qquad a_n \overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{N} x[-k]$$

$$x[n] = \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j\frac{2\pi}{N}kn} \xrightarrow[(-n)\to k]{k\to n} a_n = \frac{1}{N} \sum_{k=<N>} x[-k] e^{j\frac{2\pi}{N}kn}$$

5.7.2 Between the DTFT and CTFS

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = \sum_{m=-\infty}^{+\infty} x[-m] e^{j\omega m} \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$\Rightarrow$ Fourier series representation of $X(e^{j\omega})$

$$X(e^{jt}) = \sum_{m=-\infty}^{+\infty} x[-m] e^{jmt} = \sum_{m=-\infty}^{+\infty} x[-m] e^{jm\omega_0 t} \quad \left(\because \quad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1 \right)$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

5.8 Systems characterized by linear constant-coefficient difference equation

$$y[n] - ay[n-1] = x[n]$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \qquad h[n] = a^n u[n]$$

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

$$H(e^{j\omega})=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}$$

$$=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}=\frac{4}{\left(1-\frac{1}{2}e^{-j\omega}\right)}-\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$h[n]=4\left(\frac{1}{2}\right)^{n}u[n]-2\left(\frac{1}{4}\right)^{n}u[n]$$

Ex. 5.20)

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]\left[\frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]$$

$$= \frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

$$Y(e^{j\omega}) = \frac{B_{11}}{\left(1-\frac{1}{4}e^{-j\omega}\right)} + \frac{B_{12}}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{B_{21}}{\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$Y(e^{j\omega}) = -\frac{4}{\left(1-\frac{1}{4}e^{-j\omega}\right)} - \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]$$