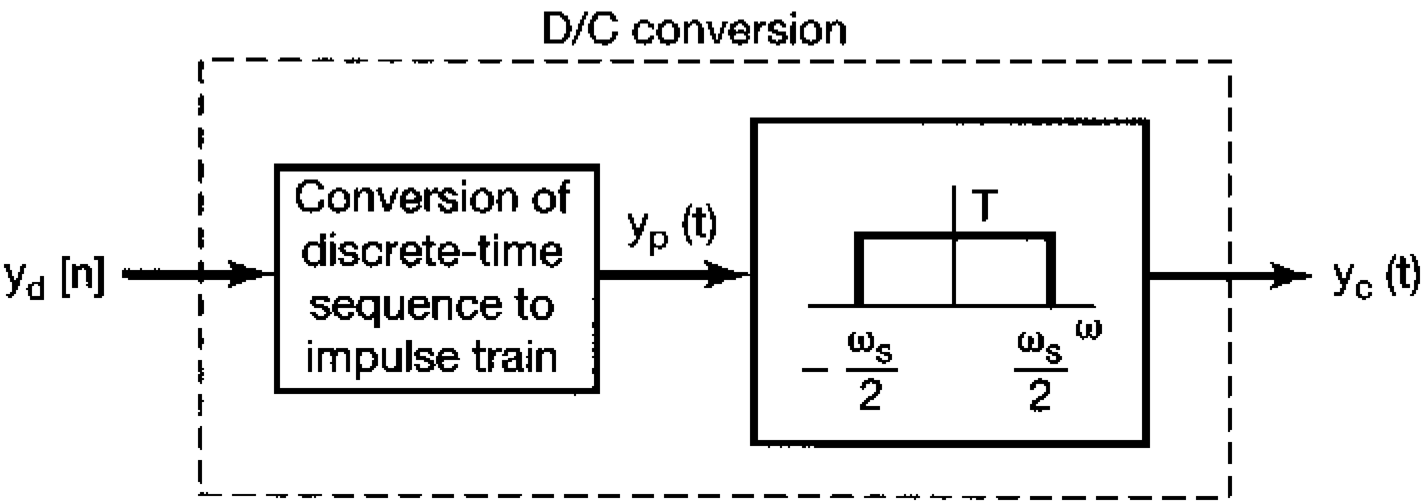
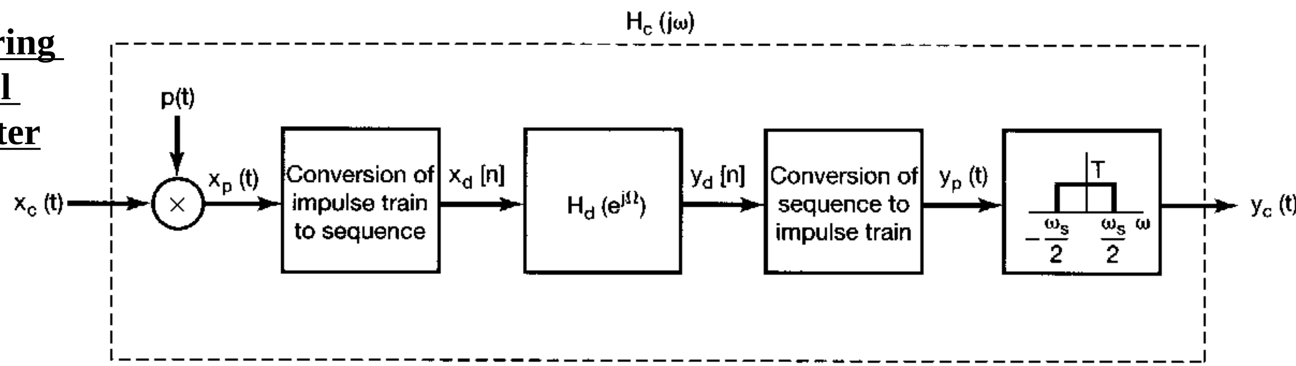
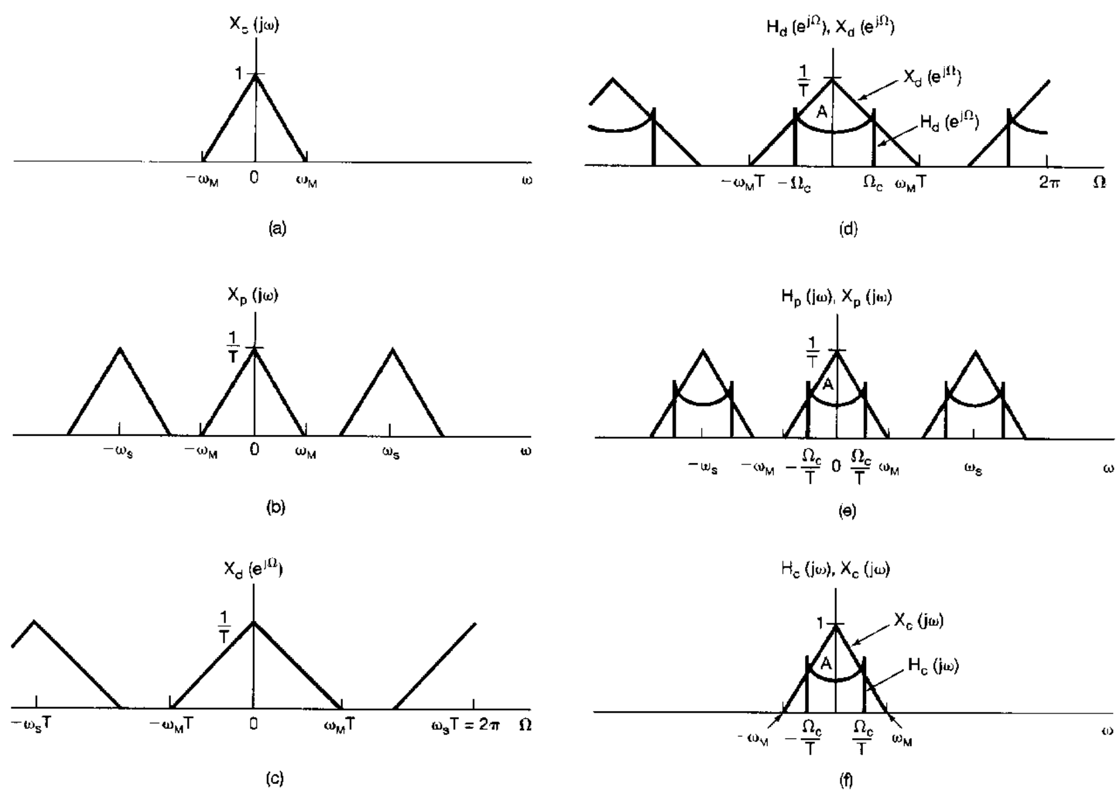
Conversion of a discrete-time sequence to a continuous-time signal



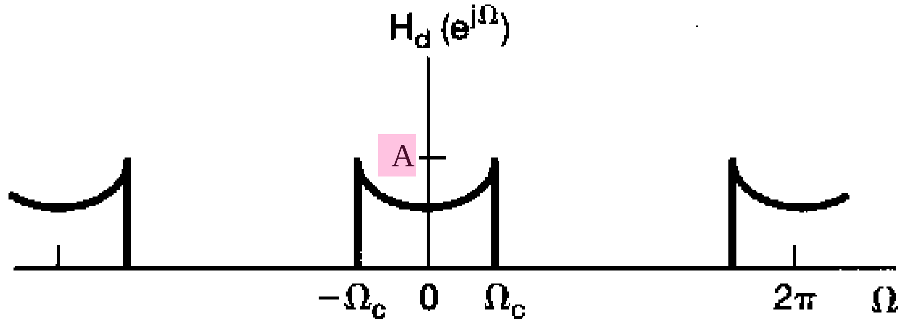
Overall system for filtering a continuous-time signal using a discrete-time filter

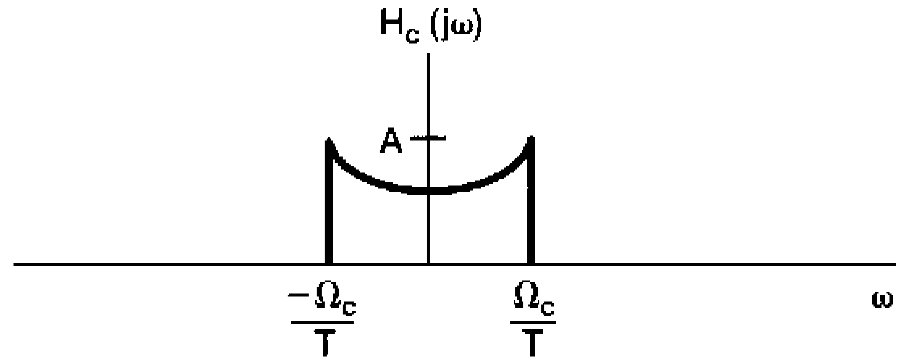




$$Y_c(j\omega) = X_c(j\omega) H_d(e^{j\omega T})$$

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s / 2 \\ 0, & |\omega| > \omega_s / 2 \end{cases}$$





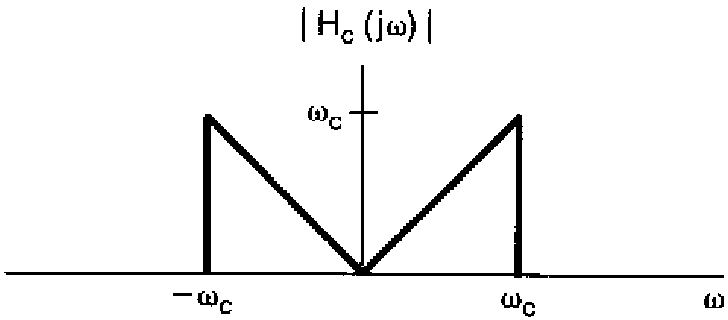
7.4.1 Digital differentiator

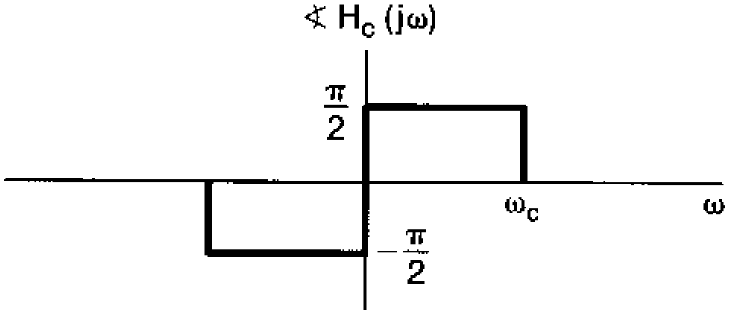
Continuous-time differentiator

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

$$H_c(j\omega) = j\omega$$

$$H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

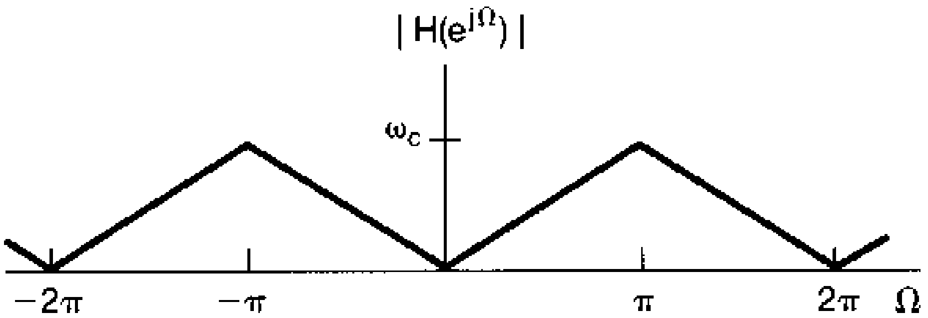




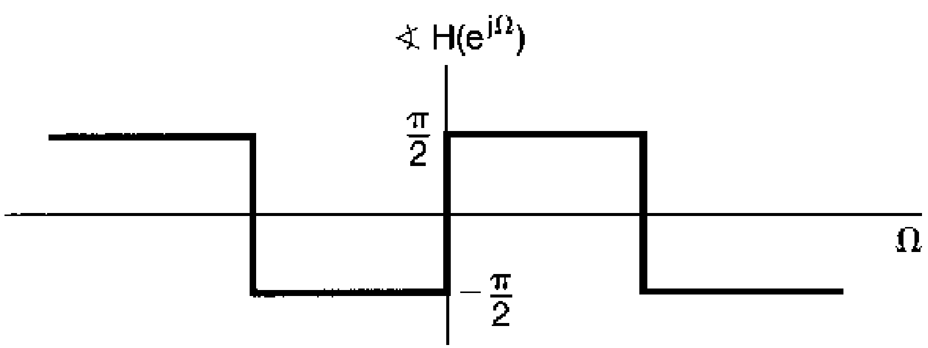
With $\quad \omega_s = 2\omega_c$

Using $H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \omega_s / 2 \\ 0, & |\omega| > \omega_s / 2 \end{cases}$

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi$$



$$h_d[n] = ?$$



Ex. 7.2)

$$x_c(t) = \frac{\sin(\pi t / T)}{\pi t} \qquad X_c(j\omega) = \begin{cases} 1, & |\omega| < \pi/T \\ 0, & \text{otherwise} \end{cases}$$

$$y_c(t) = \frac{d}{dt} x_c(t) = \frac{\cos(\pi t / T)}{Tt} - \frac{\sin(\pi t / T)}{\pi t^2}$$

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]$$: scaled unit impulse

$$y_d[n] = y_c(nT) = \begin{cases} \dfrac{(-1)^n}{nT^2}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$: impulse response scaled by $1/T$

- Digital differentiator

$$H_d(e^{j\Omega}) = j\left(\frac{\Omega}{T}\right), \quad |\Omega| < \pi$$

$\downarrow$ Impulse response

$$h_d[n] = \begin{cases} \dfrac{(-1)^n}{nT}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

- Note) Using inverse DTFT,

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} j\left(\frac{\Omega}{T}\right) e^{j\Omega n} d\Omega = \frac{j}{2\pi T}\int_{-\pi}^{+\pi} \Omega e^{j\Omega n} d\Omega$$

$$= \frac{j}{2\pi T}\cdot\frac{2\pi(-1)^n}{jn} = \frac{(-1)^n}{nT} \qquad (\text{for } n \neq 0)$$

$$\int_{-\pi}^{+\pi} \Omega e^{j\Omega n} d\Omega = \frac{1}{jn}\Omega e^{j\Omega n}\Big|_{-\pi}^{\pi} - \frac{1}{jn}\cdot\frac{1}{jn} e^{j\Omega n}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{jn}\left(\pi e^{j\pi n} + \pi e^{-j\pi n}\right) - \frac{1}{(jn)^2}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

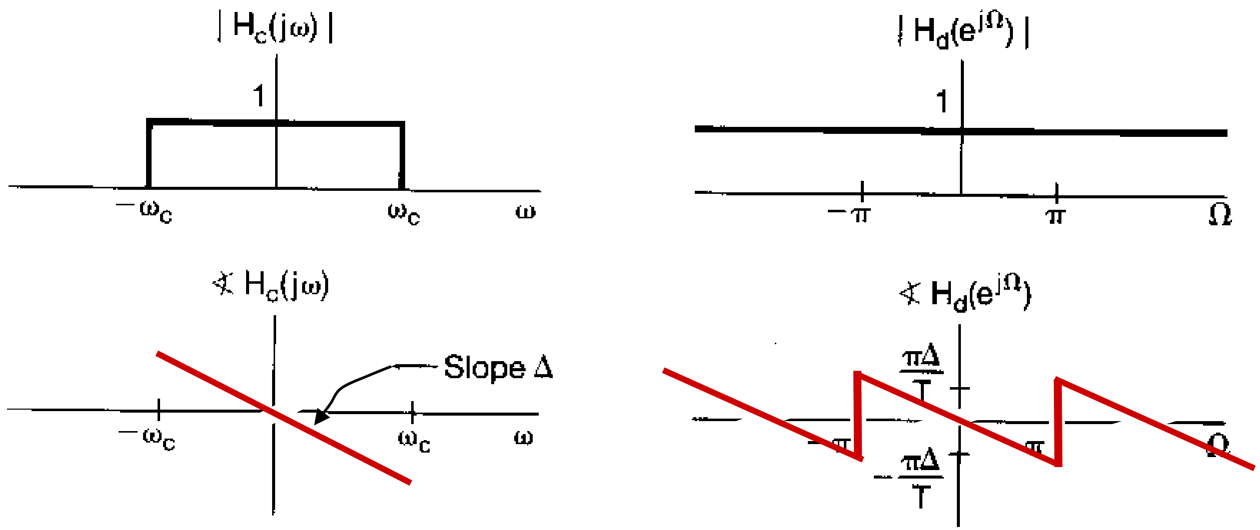
$$= \frac{2\pi(-1)^n}{jn}$$

7.4.2 Half-sample delay

$$y_c(t) = x_c(t-\Delta) \qquad\qquad Y_c(j\omega) = e^{-j\omega\Delta} X_c(j\omega)$$

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & otherwise \end{cases}$$

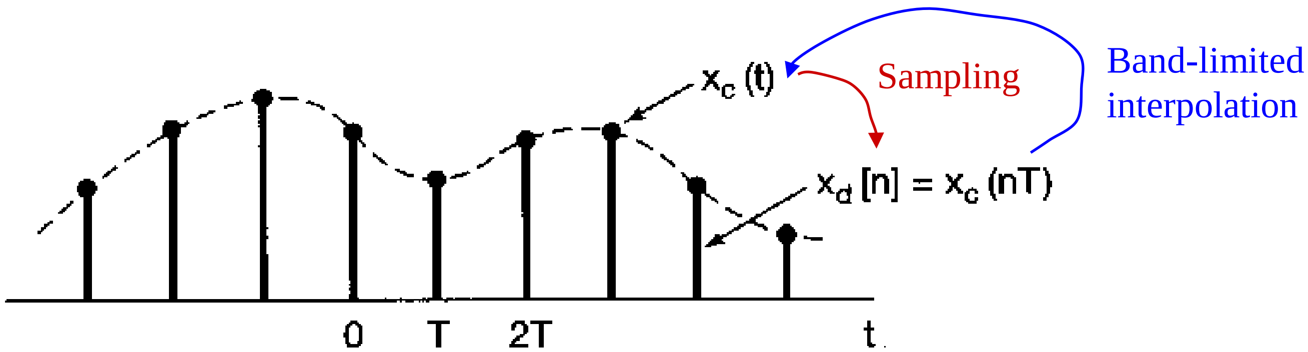
With $\quad \omega_s = 2\omega_c \qquad\qquad H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi$



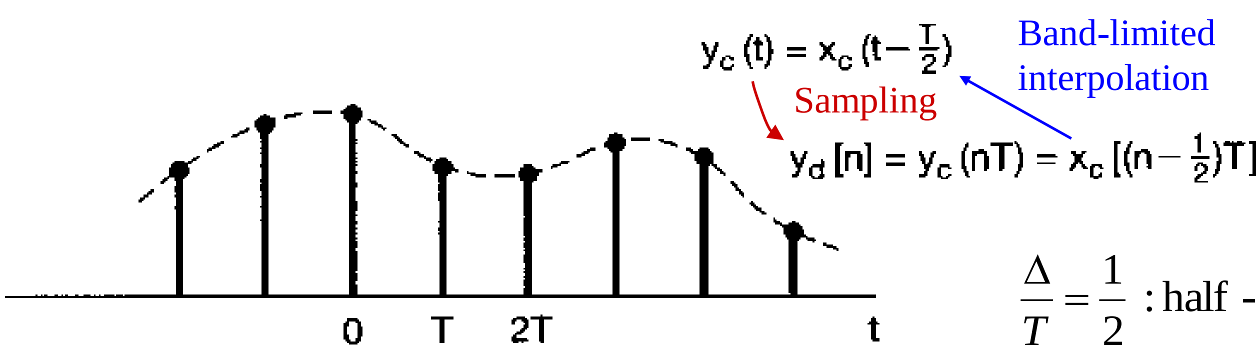
(a) (b)

Interpretation of a non-integer delay

$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right] \Rightarrow \text{samples of a shifted version of the band-limited interpolation of } x_d[n]$$



(a)



(b)

$$\frac{\Delta}{T} = \frac{1}{2} \text{ : half-sample delay}$$

Ex. 7.3) Determine the impulse response $h_d[n]$ of the discrete-time filter in the half-sample delay system

$$x_c(t) = \frac{\sin(\pi t / T)}{\pi t}$$

$$x_d[n] = x_c(nT) = \frac{1}{T}\delta[n]$$: scaled unit impulse

$$y_c(t) = x_c(t - T/2) = \frac{\sin(\pi(t - T/2)/T)}{\pi(t - T/2)}$$

$$y_d[n] = y_c(nT) = \frac{\sin(\pi(n - \frac{1}{2}))}{T\pi(n - \frac{1}{2})}$$: impulse response scaled by $1/T$

$$\therefore \quad h[n] = \frac{\sin(\pi(n - \frac{1}{2}))}{\pi(n - \frac{1}{2})}$$

- Note) Using inverse DTFT,

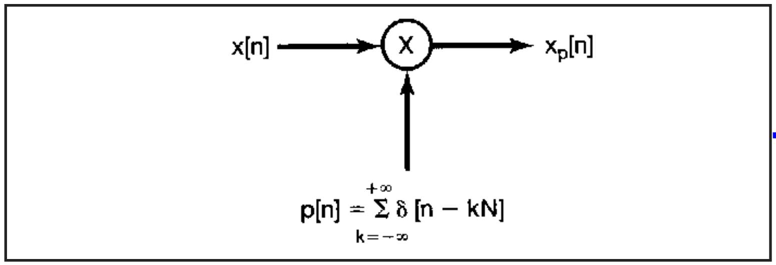
$$H_{d,1}(e^{j\Omega}) = 1, \quad |\Omega| < \pi \leftrightarrow h_{d,1}[n] = \frac{\sin \pi n}{\pi n}$$

$$H_d(e^{j\Omega}) = 1 \cdot e^{-j\Omega\left(\frac{1}{2}\right)}, \quad |\Omega| < \pi \leftrightarrow h_d[n] = \frac{\sin \pi\left(n - \frac{1}{2}\right)}{\pi\left(n - \frac{1}{2}\right)}$$

7.5 SAMPLING OF DISCRETE-TIME SIGNALS

7.5.1 Impulse-train sampling

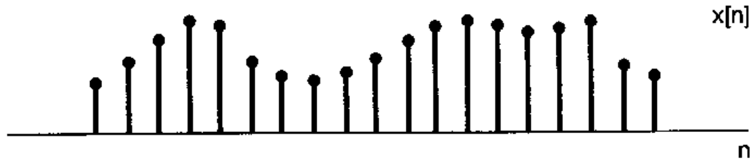
$$x_p[n] = \begin{cases} x[n], & \text{if } n = \text{an integer multiple of } N \\ 0, & \text{otherwise} \end{cases}$$



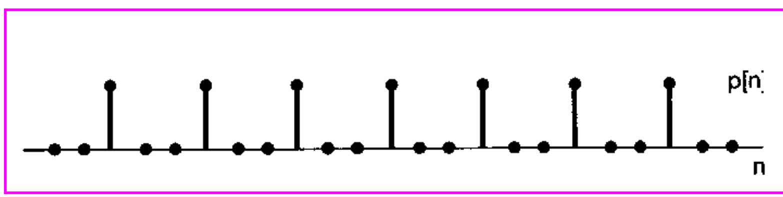
$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n-kN]$$

Multiplication property

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

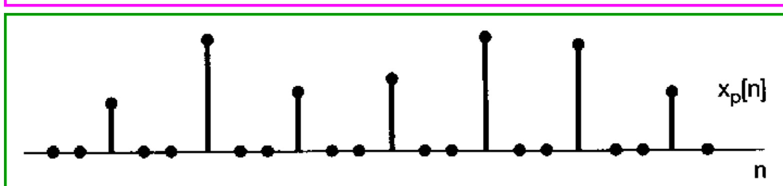


$$\omega_s = \frac{2\pi}{N}$$



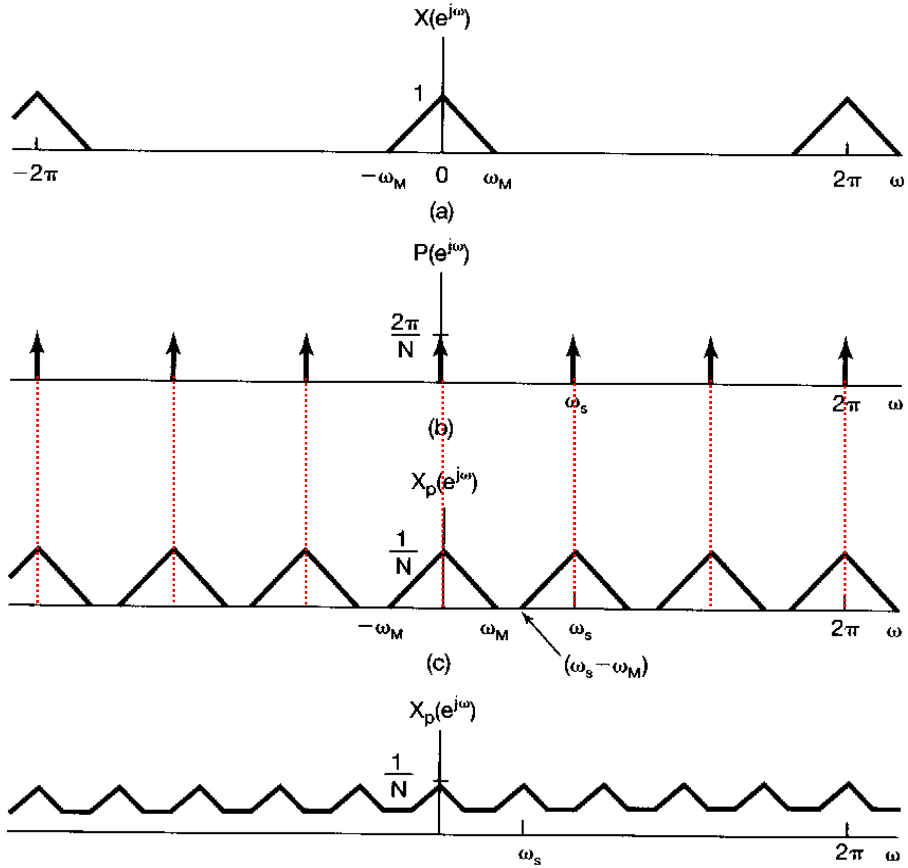
DTFT

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s)$$



DTFT

$$\boxed{X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})}$$

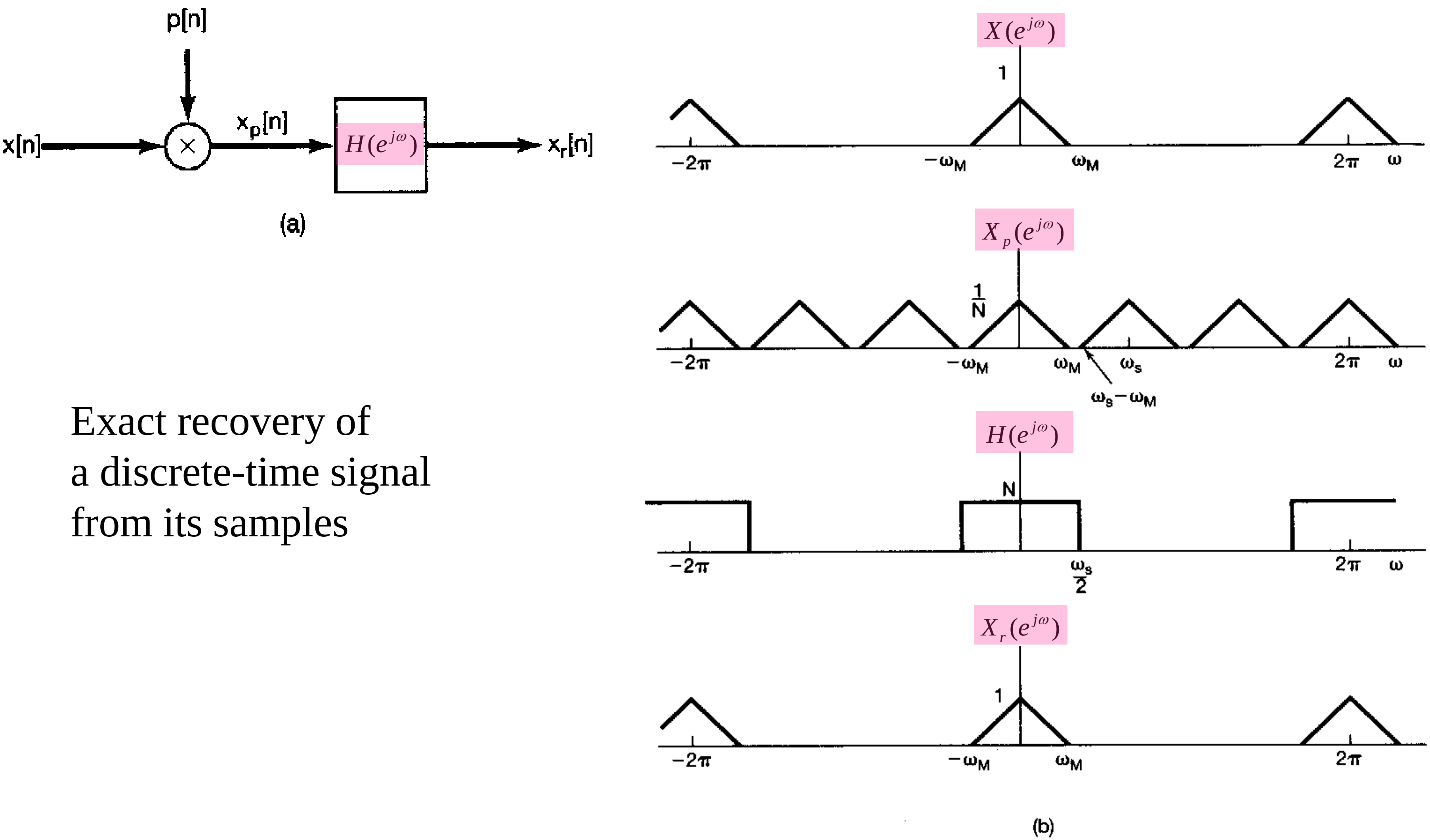


$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\omega_s)$$

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$

Aliasing $(\omega_s < 2\omega_M)$

- Note) Irrespective of aliasing, $x_r[kN] = x[kN], \quad k = 0, \pm 1, \pm 2, \ldots$
- Note) In the sampling of DT signals, $\Omega = \omega T$? **No**



Exact recovery of
a discrete-time signal
from its samples

7.5.2 Discrete-time decimation and interpolation

Decimation (sub-sampling, downsampling)

$$x_b[n] = x_p[nN] \qquad x_b[n] = x[nN]$$

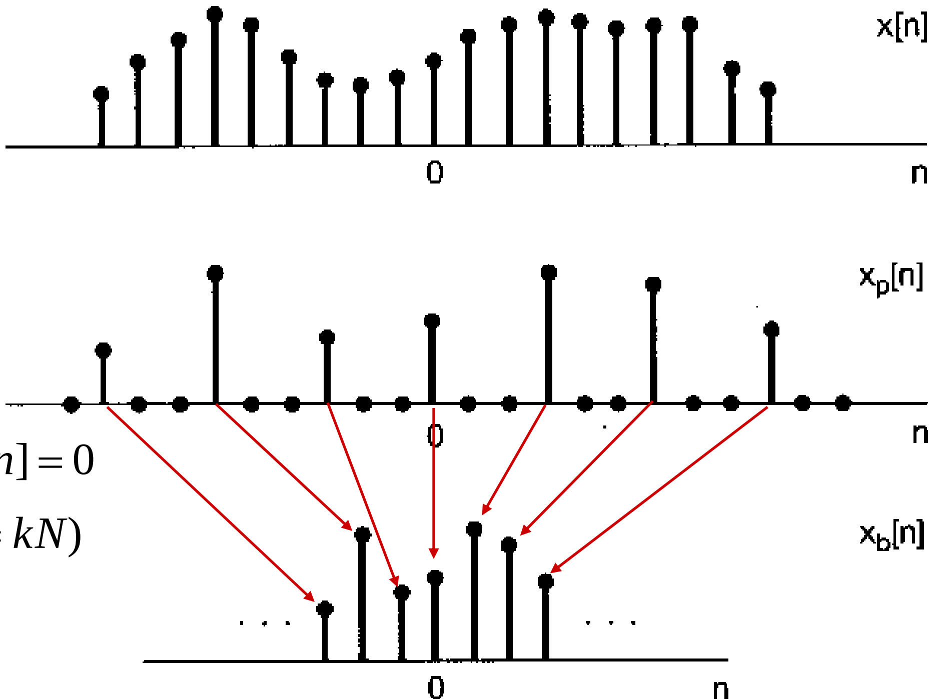
$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k] e^{-j\omega k}$$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_p[kN] e^{-j\omega k}$$

$n = kN$

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_p[n] e^{-j\omega n/N} \quad (\because x_p[n] = 0 \text{ for } n \neq kN)$$

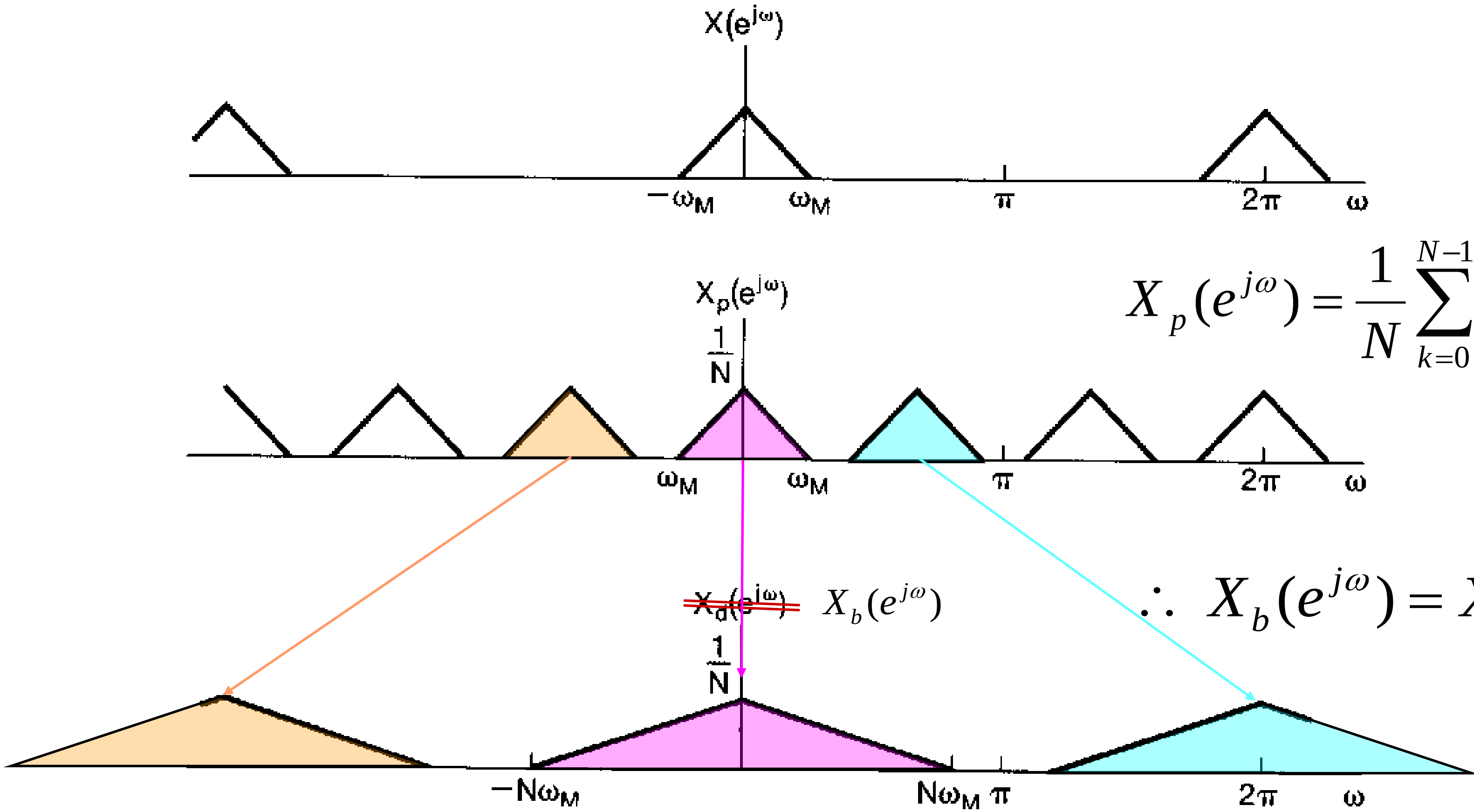
$$= X_p(e^{j\omega/N})$$



$$\boxed{\therefore \; X_b(e^{j\omega}) = X_p(e^{j\omega/N})}$$

Frequency scaling (or normalization)

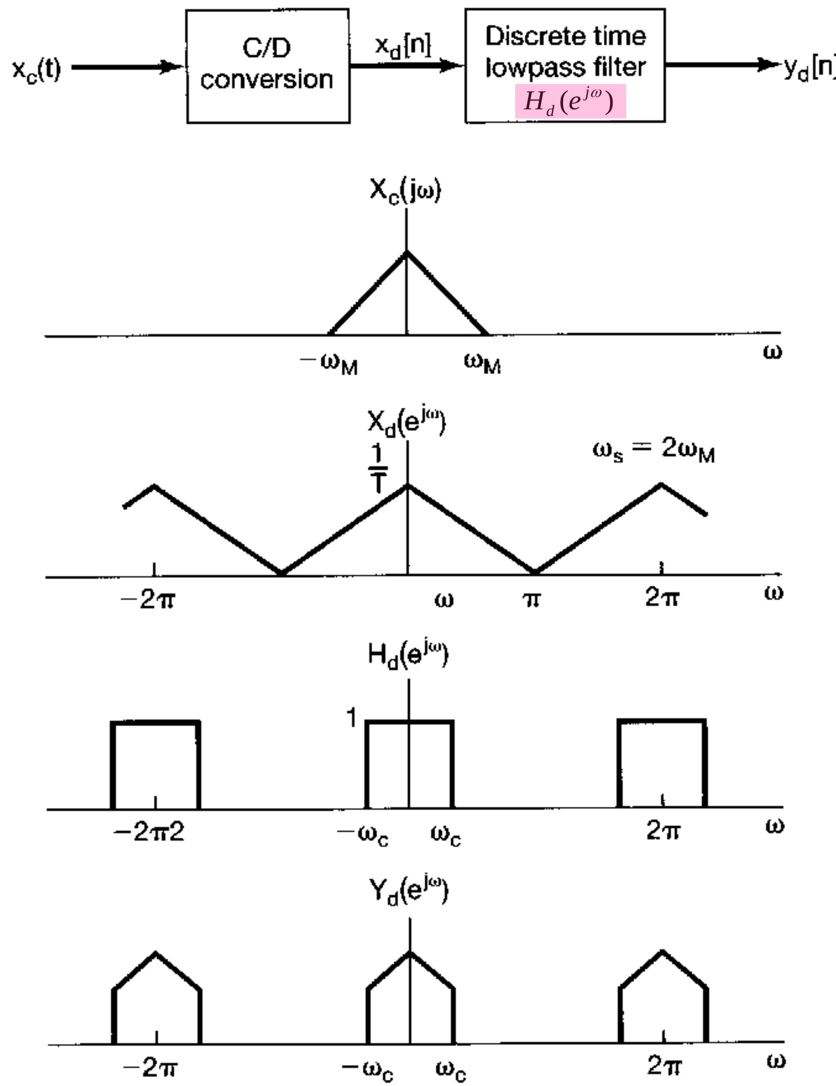
The relation between sampling and decimation



$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$

$$\therefore \ X_b(e^{j\omega}) = X_p(e^{j\omega/N})$$

Application example



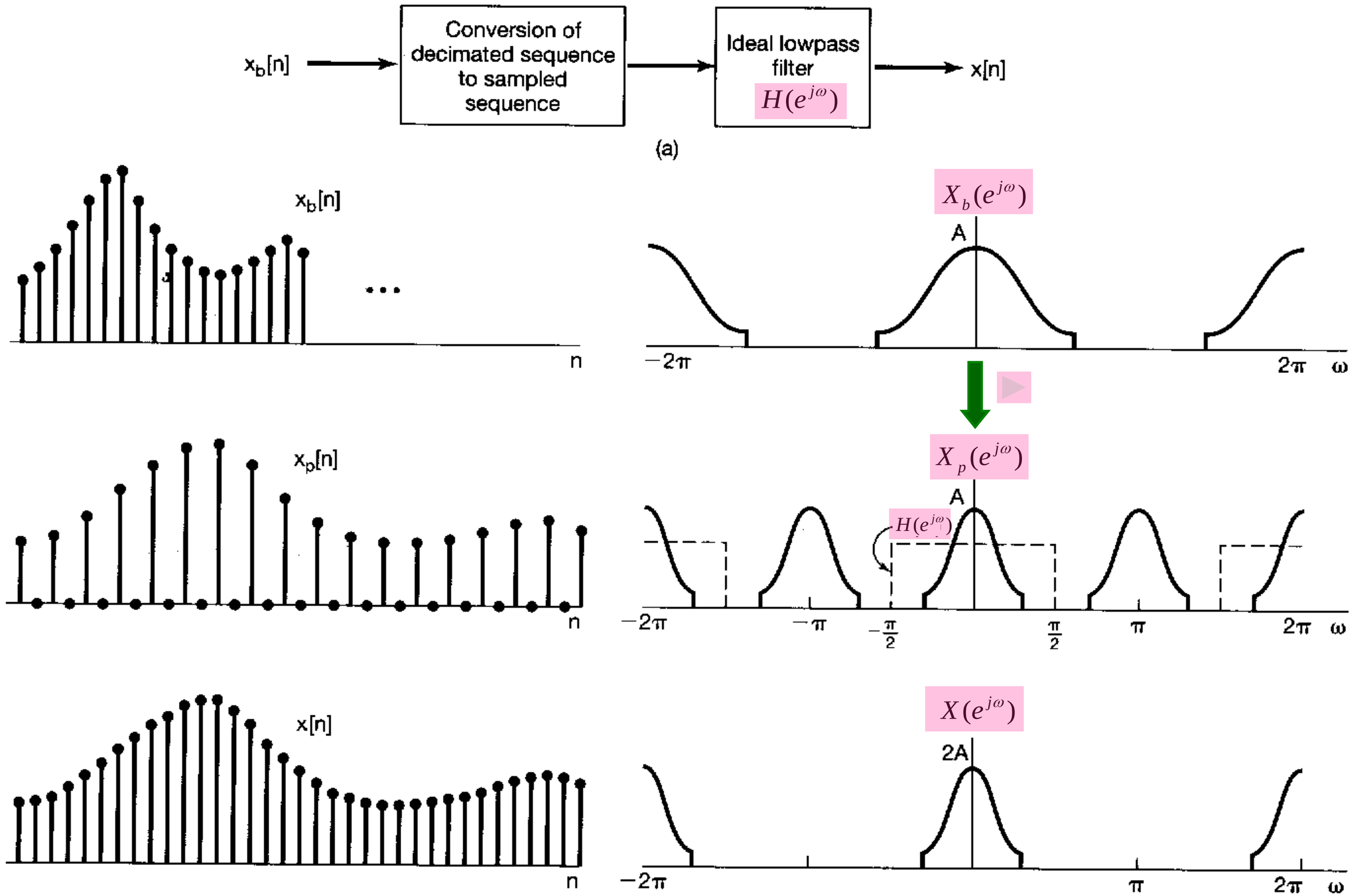
Sampled at the Nyquist rate

Discrete-time LPF

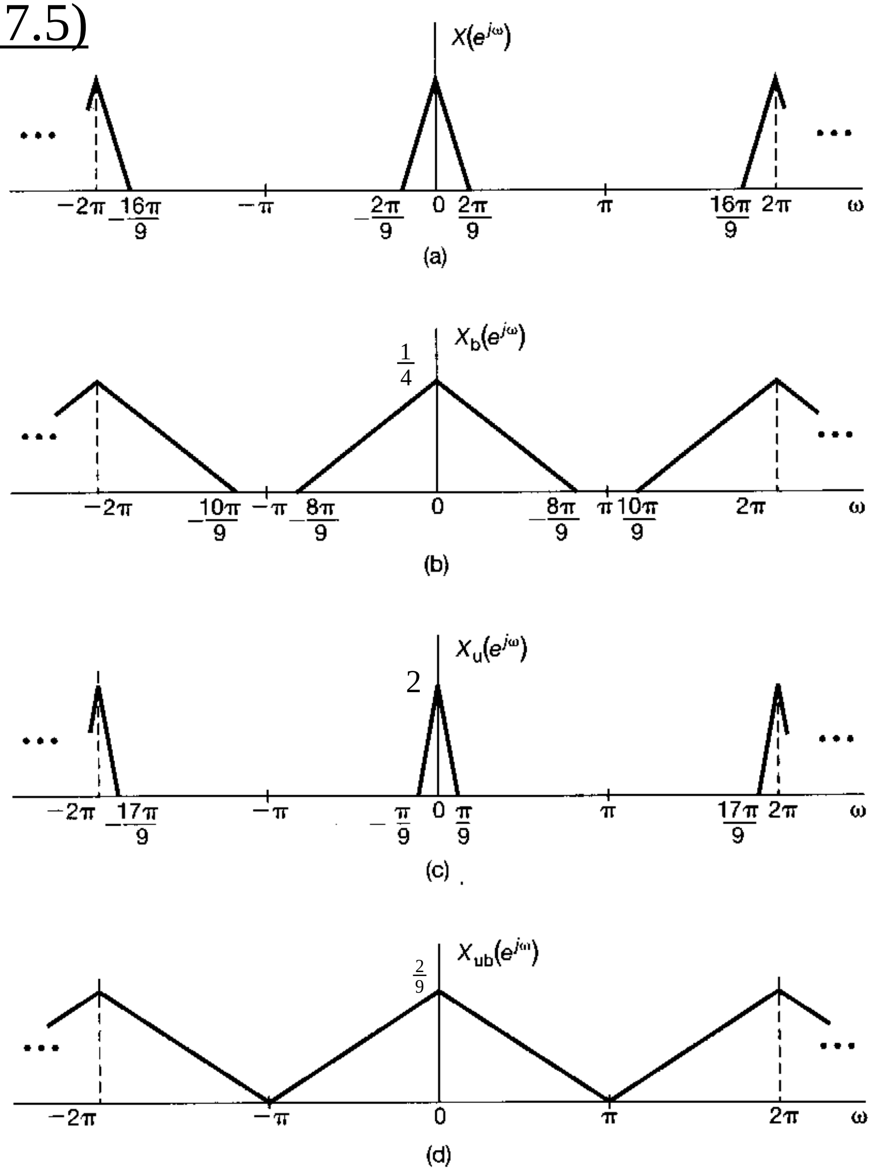
Band-limited output

$\Rightarrow$ downsampling or decimation can be applied

Interpolation (Upsampling): the reverse of decimation or downsampling



Ex. 7.5)



downsampled by 4

upsampling $x[n]$ by a factor of 2

upsampling $x[n]$ by 2
and then downsampling by 9