10

The *z*-Transform

10. The *z*-Transform

Laplace transform - for continuous time signal/system
z-transform - for discrete time signal/system

10.1 The *z*-transform

For an input $x[n] = z^n$

$$y[n] = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

↓ $z = e^{j\omega}$ with ω real (i.e., $|z| = 1$)

DTFT of $h[n]$

z-transform of $h[n]$ (when $|z|$ is not restricted to unity)

- z-transform of a general discrete-time signal $x[n]$

$$X(z) \stackrel{\Delta}{=} \sum_{n=-\infty}^{+\infty} x[n] z^{-n}$$

- Notation : $x[n] \overset{z}{\longleftrightarrow} X(z)$

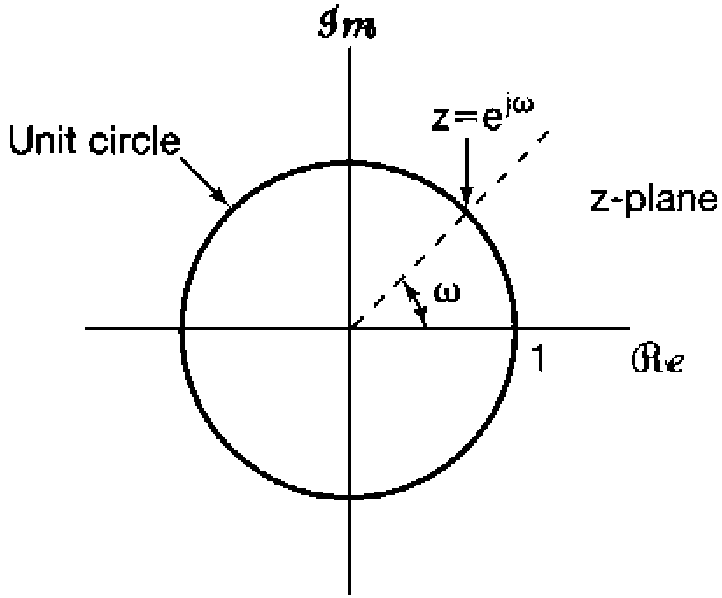
For a complex variable z, let $z = re^{j\omega}$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n}$$

$$X(re^{j\omega}) = \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\} e^{-j\omega n}$$

$$X(re^{j\omega}) = \Im\{x[n]r^{-n}\}$$

$|z| = 1 = |e^{j\omega}|$ is equivalent to the Fourier Transform



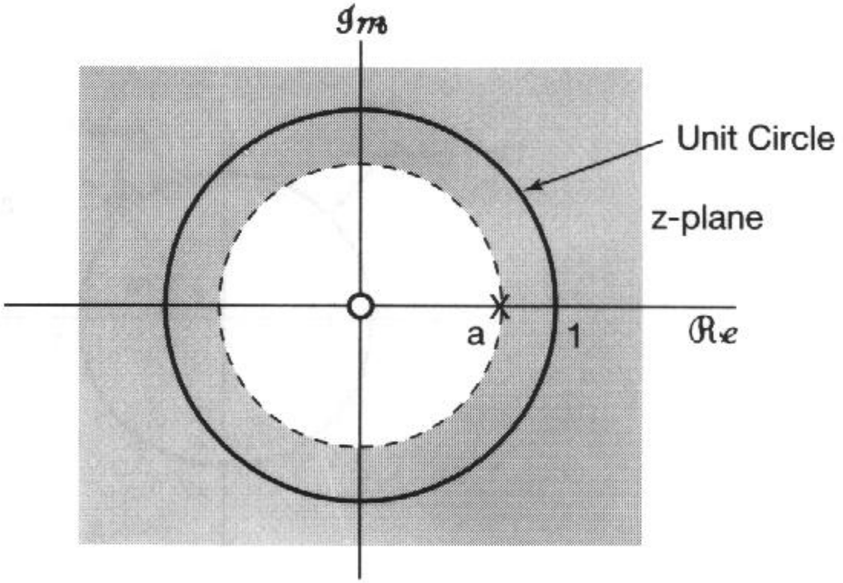
- Region of convergence (ROC) : a range of values of z for which $X(z)$ converges.

Note) If the ROC includes the unit circle, then the Fourier transform also converges.

Ex. 10.1) $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \qquad |z| > |a|$$

Region of Convergence (ROC)	$\lvert az^{-1} \rvert < 1$



Pole-zero plot and region of convergence for $0 < a < 1$

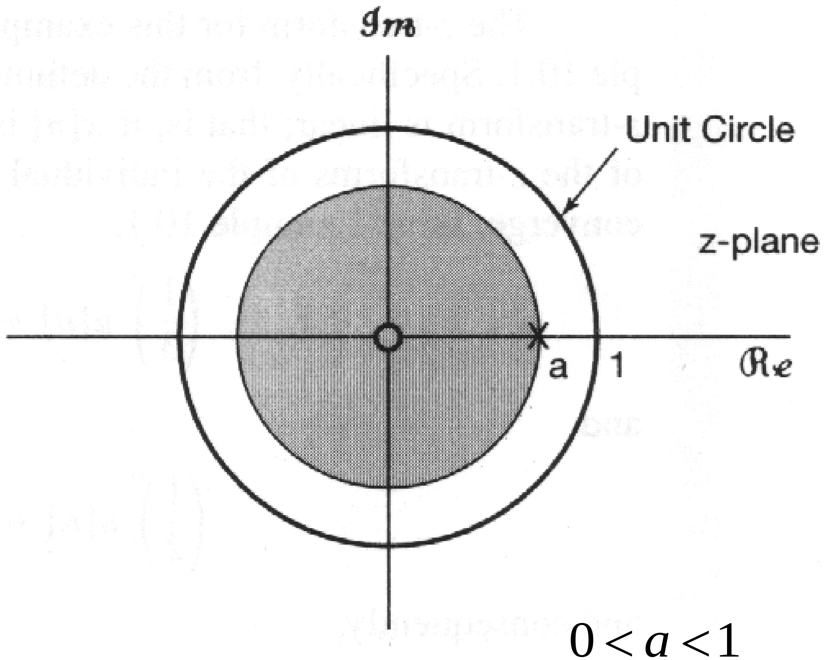
Ex.10.2) $x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

$$X(z) = 1 - \frac{1}{1 - a^{-1} z}$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| \langle |a|$$



$0 < a < 1$

Ex. 10.3) $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

$$X(z) = \sum_{n=-\infty}^{+\infty} \left\{ 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \right\} z^{-n}$$

$$= 7\sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u[n] z^{-n} - 6\sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n}$$

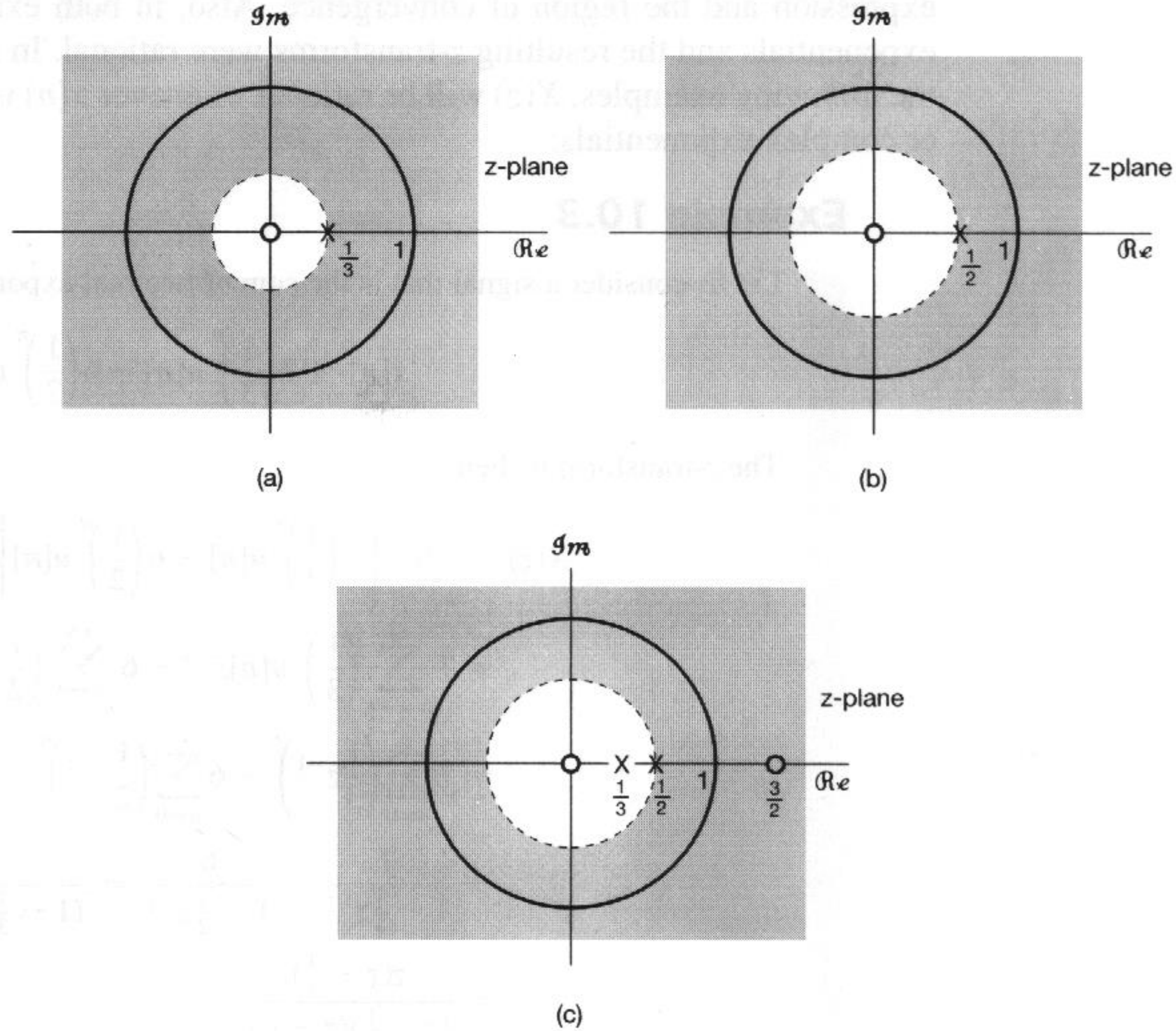
$$= 7\sum_{n=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^n - 6\sum_{n=0}^{+\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

$$= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}$$

ROC? $\left|\frac{1}{3}z^{-1}\right| < 1$ & $\left|\frac{1}{2}z^{-1}\right| < 1$

$|z| > \frac{1}{3}$ & $|z| > \frac{1}{2}$



Im
z-plane
Re
1/3
1
(a)
Im
z-plane
Re
1/2
1
(b)
Im
z-plane
Re
1/3
1/2
1
3/2
(c)

Ex. 10.4) $x[n]=\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n]$

$$=\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n]-\frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]$$

$$X(z)=\sum_{n=-\infty}^{\infty}\left\{\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n]-\frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]\right\}z^{-n}$$

$$=\frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{j\pi/4}z^{-1}\right)^n-\frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\pi/4}z^{-1}\right)^n$$

$$=\frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}}-\frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}}$$

ROC는?

$$\left|\frac{1}{3}e^{j\pi/4}z^{-1}\right|<1 \;\; \& \;\; \left|\frac{1}{3}e^{-j\pi/4}z^{-1}\right|<1$$

$$\therefore |z|>\frac{1}{3}$$

$$X(z)=\frac{\frac{1}{3\sqrt{2}}z}{\left(z-\frac{1}{3}e^{j\pi/4}\right)\left(z-\frac{1}{3}e^{-j\pi/4}\right)}$$

10.2 The Region of Convergence for the *z*-Transform

Property 1 : The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

$$z = re^{j\omega} \Rightarrow \sum_{n=-\infty}^{+\infty} |x[n]| r^{-n} < \infty \quad \text{(Fig.10.6)}$$

Property 2 : The ROC does not contain any poles.

why ? $X(z)$ is infinite at a pole

Property 3 : If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly z=0 and/or $z=\infty$.

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n},$$

case1) $N_1 : -,\ N_2 : + \quad z = 0 \ \ or \ \ z = \infty \ \not\subset ROC$

case2) $N_1 \geq 0 \quad z = \infty \ \subset ROC$

case3) $N_2 \leq 0 \quad z = 0 \ \subset ROC$

Ex.) : finite sequence {1,2,3}

$$X(z) = 1 + 2z^{-1} + 3z^{-2}$$

ROC : entire complex plane except for z = 0.

$$X(z) = z + 2 + 3z^{-1}$$

ROC : entire z-plane except for z = 0 and $z = \infty$.

$$X(z) = z^4 + 2z^3 + 3z^2$$

ROC : entire z-plane except for $z = \infty$.

Property 4 : If $x[n]$ is a right-sided sequence, and if the circle $|z|=r_0$ is in the ROC, then all finite values of z for which $|z|>r_0$ will also be in the ROC.

$$X(z)=\sum_{n=N_1}^{\infty} x[n]z^{-n} \ (N_1 : \text{positive or negative or } 0)$$

$$|z|=r_1 \text{ with } r_1>r_0 \Rightarrow x[n]r_1^{-n} \text{ is abs. summable if } x[n]r_0^{-n} \text{ is also for } N_1 \le n < \infty.$$

Property 5 : If $x[n]$ is a left-sided sequence, and if the circle $|z|=r_0$ is in the ROC, then all values of z for which $0<|z|<r_0$ will also be in the ROC.

$$X(z)=\sum_{n=-\infty}^{N_2} x[n]z^{-n} \ (N_2 : \text{positive or negative or } 0)$$

$$|z|=r_2 \text{ with } r_2<r_0 \Rightarrow x[n]r_2^{-n} \text{ is abs. summable if } x[n]r_0^{-n} \text{ is also for } -\infty < n \le N_2.$$

Property 6 : If $x[n]$ is two sided, and if the circle $|z|=r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|=r_0$

refer to Fig.10.8, p.752

Ex. 10.7) $x[n] = b^{|n|}, \quad b > 0$ (Fig.10.10, p753)

$$= b^n u[n] + b^{-n} u[-n-1]$$

$$b^n u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1 - bz^{-1}}, |z| > b$$

$$b^{-n} u[-n-1] \stackrel{z}{\longleftrightarrow} \frac{-1}{1 - b^{-1} z^{-1}}, |z| < \frac{1}{b}$$

ROC $\Rightarrow$ Fig.10.11(a)–(d) (p.755)

For $b>1$, (a), (b) $\Rightarrow$ no common ROC

For $b<1$, (c), (d) $X(z) = \frac{1}{1 - bz^{-1}} - \frac{1}{1 - b^{-1}z^{-1}}, b < |z| < \frac{1}{b}$

$$= \frac{b^2 - 1}{b} \frac{z}{(z-b)(z-b^{-1})}, b < |z| < \frac{1}{b}$$

(ROC : Fig. 10.11(e) at p.755)

Property 7 : If the z-transform $X(z)$ of $x[n]$ is rational,
then its ROC is bounded by poles or extends to infinity

Property 8 : $X(z)$: rational, $x[n]$: right sided
=> ROC : the region in the z-plane outside the outermost pole.

Note) if $x[n]$ is causal, ROC ? includes $z = \infty$

Property 9 : $X(z)$: rational, $x[n]$: left sided
=> ROC : the region in the z-plane inside the innermost nonzero pole.

Note) if $x[n]$ is anticausal, ROC ? includes $z = 0$

10.3 The Inverse z-Transform

$$X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\} \Leftrightarrow x[n]r^{-n} = \mathcal{F}^{-1}\{X(re^{j\omega})\}$$

$$x[n] = r^n \mathcal{F}^{-1}\left[X(re^{j\omega})\right] = r^n \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega$$

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

$$z = re^{j\omega}$$

$$dz = jre^{j\omega}d\omega = jzd\omega$$

$$\therefore d\omega = (1/j)z^{-1}dz$$

From the Residue theorem

$$x[n] = \mathrm{Res}\{X(z)z^{n-1}\}$$

$\oint$: integration around a counter clockwise closed circular contour centered at the origin and with radius r

- Alternative method for the inverse z-transform

  : partial-fraction expansion

Ex. 10.9 - 11)

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}, \qquad |z|>\tfrac{1}{3}$$

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{2}{1-\frac{1}{3}z^{-1}} \Longrightarrow x[n]=\left(\frac{1}{4}\right)^n u[n]+2\left(\frac{1}{3}\right)^n u[n]$$

ROC : (1/4)<|z|<(1/3) $\Rightarrow$ $x[n]=\left(\frac{1}{4}\right)^n u[n]-2\left(\frac{1}{3}\right)^n u[-n-1]$

ROC : |z|<(1/4) $\Rightarrow$ $x[n]=-\left(\frac{1}{4}\right)^n u[-n-1]-2\left(\frac{1}{3}\right)^n u[-n-1]$

• Another procedure for the inverse *z*-transform

: power-series expansion of *X*(*z*)

Ex. 10.12)

$$X(z) = 4z^{2} + 2 + 3z^{-1}, \qquad 0 \langle |z| \langle \infty$$

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

Ex. 10.13)

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$= 1 + az^{-1} + a^2 z^{-2} + \cdots \quad \text{(by long division)}$$

$$\therefore x[n] = a^n u[n]$$

If $|z| < |a|$,

$$\frac{1}{1-az^{-1}} = \frac{1}{(-az^{-1})(1-a^{-1}z)} = (-a^{-1}z)(1+a^{-1}z+(a^{-1}z)^2 + \cdots)$$

$$= -a^{-1}z - a^{-2}z^2 - \cdots$$

$$\therefore x[n] = -a^n u[-n-1]$$

10.4 Geometric Evaluation of the Fourier Transform From the Pole-Zero Plot

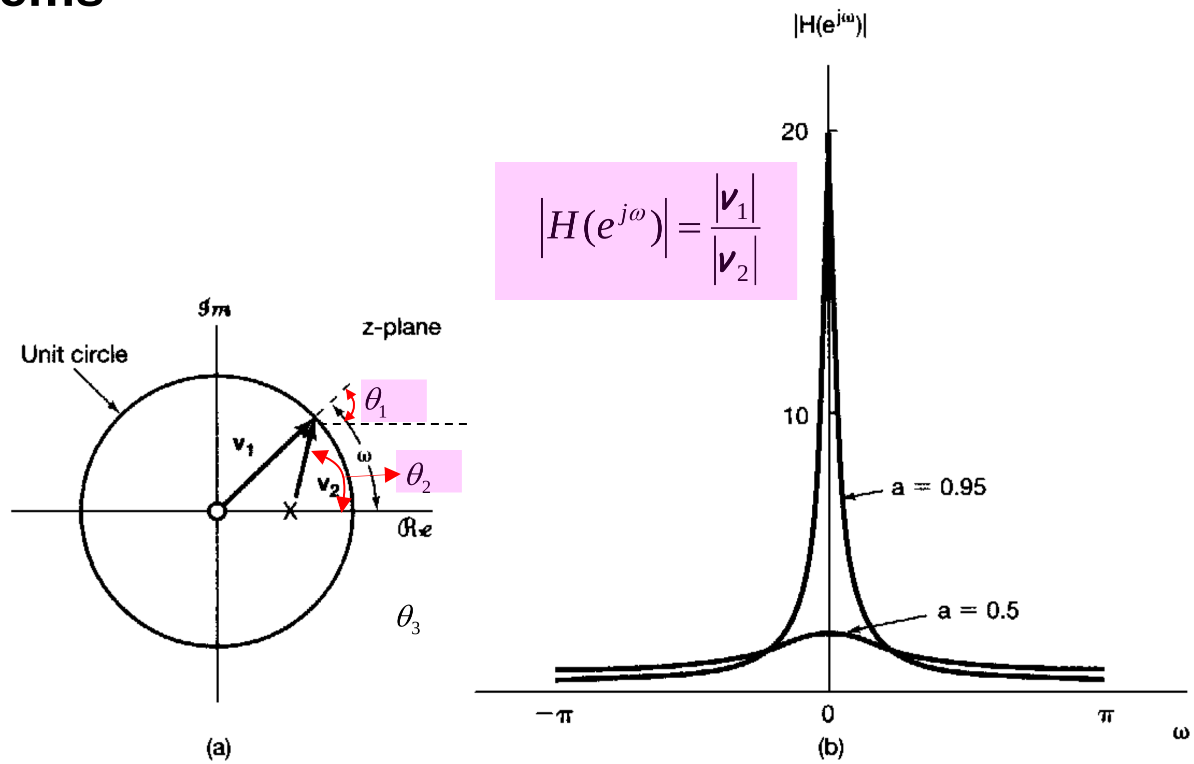
10.4.1 First-Order Systems

$$h[n] = a^n u[n]$$

$$H(z) = \frac{1}{1 - az^{-1}}$$

$$= \frac{z}{z - a} \qquad |z| > |a|$$

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



$$|H(e^{j\omega})| = \frac{|\mathbf{v}_1|}{|\mathbf{v}_2|}$$

$$\angle H(e^{j\omega}) = \theta_1 - \theta_2$$

(Fig. 10.13, p. 764)

10.4.2 Second-Order Systems

$$h[n] = r^n \frac{\sin(n+1)\theta}{\sin\theta} u[n] \quad \& \quad H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}$$

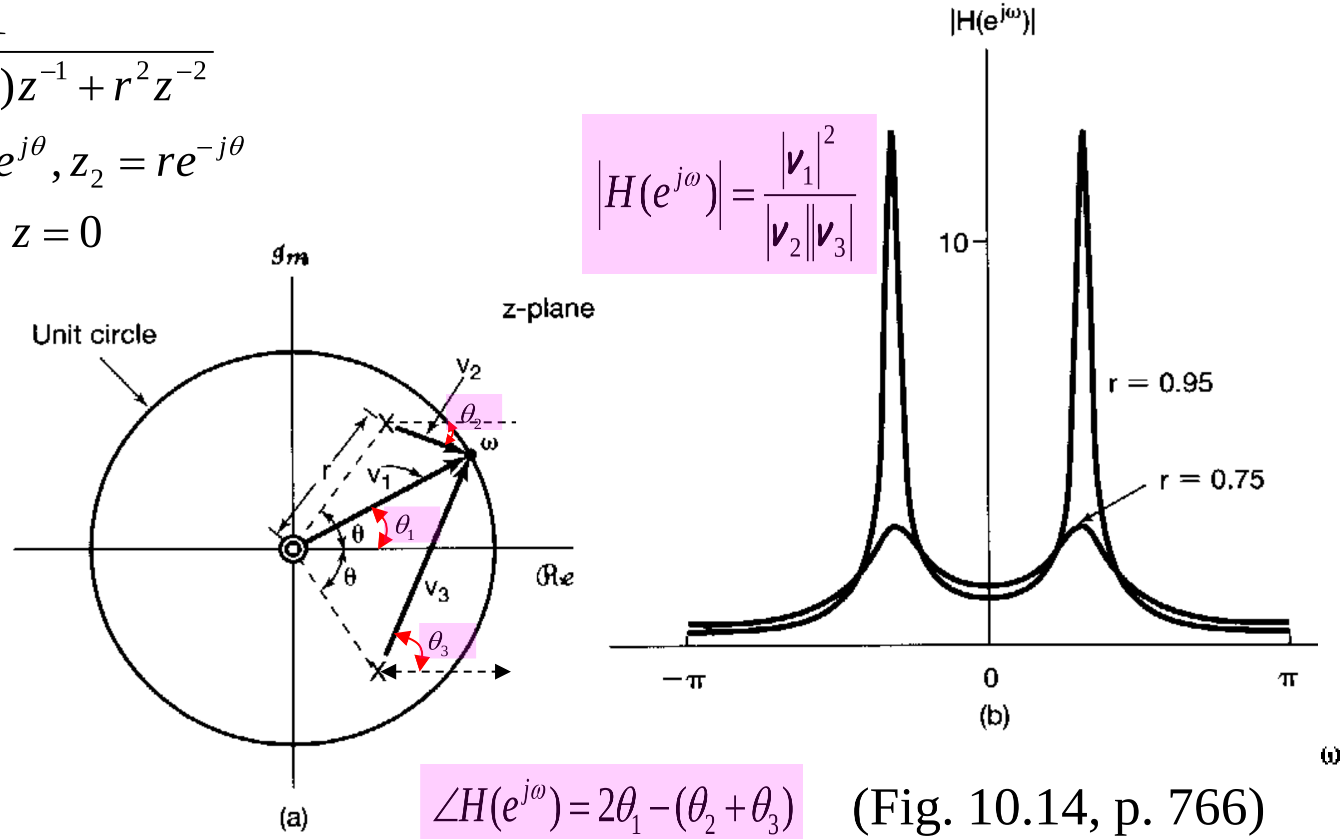
$$(0 < r < 1 \quad \& \quad 0 \le \theta \le \pi)$$

$$\Rightarrow H(z) = \frac{1}{1 - (2r\cos\theta)z^{-1} + r^2 z^{-2}}$$

· Pole locations : $z_1 = re^{j\theta}, z_2 = re^{-j\theta}$

· zero : double zeros at $z = 0$

$$|H(e^{j\omega})| = \frac{|v_1|^2}{|v_2||v_3|}$$



$$\angle H(e^{j\omega}) = 2\theta_1 - (\theta_2 + \theta_3)$$

(Fig. 10.14, p. 766)