10.5 Properties of the *z*-Transform

10.5.1 Linearity

$$\begin{cases} x_1[n] \xleftrightarrow{z} X_1(z), \text{ with ROC} = R_1 \\ x_2[n] \xleftrightarrow{z} X_2(z), \text{ with ROC} = R_2 \end{cases}$$

$$\Rightarrow ax_1[n] + bx_2[n] \xleftrightarrow{z} aX_1(z) + bX_2(z), \text{ with ROC contaning } R_1 \cap R_2$$

10.5.2 Time Shifting

$$x[n] \xleftrightarrow{z} X(z), \text{ with ROC} = R$$

$$\Rightarrow x[n-n_0] \xleftrightarrow{z} z^{-n_0} X(z), \text{ with ROC} = R(\text{expect for } 0 \text{ or } \infty)$$

10.5.3 Scaling in the *z*-Domain

$$x[n] \xleftrightarrow{z} X(z), \text{ with ROC} = R$$

$$\Rightarrow z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right), \text{ with ROC} = |z_0| R$$

$\cdot$ Special case : $z_0 = e^{j\omega_0}$

$$e^{j\omega_0 n} x[n] \xleftrightarrow{z} X(e^{-j\omega_0} z) \qquad (\text{Fig.}10.15)$$

10.5.4 Time Reversal

$$x[n] \xleftrightarrow{z} X(z), \ \text{with ROC} = R$$
$$\Rightarrow x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right), \ \text{with ROC} = \frac{1}{R}$$

10.5.5 Time Expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x[n] \xleftrightarrow{z} X(z), \ \text{with ROC} = R$$
$$\Rightarrow x_{(k)}[n] \xleftrightarrow{z} X(z^k), \ \text{with ROC} = R^{1/k}$$

10.5.6 Conjugation

$$x[n] \overset{z}{\longleftrightarrow} X(z), \text{ with ROC} = R$$

$$\Rightarrow x^*[n] \overset{z}{\longleftrightarrow} X^*\left(z^*\right), \text{ with ROC} = R$$

$$\text{Note) } x[n] \text{ is real } \Rightarrow X(z) = X^*(z^*)$$

$$\therefore \text{If } X(z) \text{ has a ploe (or zero) at } z = z_0, \quad \text{at } z = z_0^* ?$$

10.5.7 The Convolution Property

$$x_1[n] \overset{z}{\longleftrightarrow} X_1(z), \text{ with ROC} = R_1$$

$$x_2[n] \overset{z}{\longleftrightarrow} X_2(z), \text{ with ROC} = R_2$$

$$\Rightarrow x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z), \text{ with ROC contaning } R_1 \cap R_2$$

(Derivation : Problem 10.56)

10.5.8 Differentiation in the *z*-Domain

$$x[n] \xleftrightarrow{z} X(z), \text{ with ROC} = R$$

$$\Rightarrow nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz}, \text{ with ROC} = R$$

Ex. 10.27) $X(z) = \log(1+az^{-1}), \quad |z| > |a|$

$$nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|$$

From $\quad a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}}, \quad |z| > |a| \quad$ (Example 10.1)

$$a(-a)^n u[n] \xleftrightarrow{z} \frac{a}{1+az^{-1}}, \quad |z| > |a| \quad \text{(Linearity)}$$

$$a(-a)^{n-1} u[n-1] \xleftrightarrow{z} \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a| \quad \text{(Time shifting)}$$

$$\therefore x[n] = \frac{-(-a)^n}{n} u[n-1]$$

10.5.9 The Initial-Value Theorem

Causal Sequence

$$x[n]=0, n<0 \Rightarrow x[0]=\lim_{z\to\infty} X(z)$$

pf) $X(z)=\sum_{n=0}^{\infty} x[n]z^{-n}$ for causal $x[n]$

$$z\to\infty, \begin{cases} z^{-n}\to 0 & \text{for } n>0 \\ z^{-n}=1 & \text{for } n=0 \end{cases}$$

$\mathrm{O}(\text{numerator}) \le \mathrm{O}(\text{denominator})$

Note) For a causal $x[n]$, $x[0]$: finite

\# of finite zeros ≤ # of finite poles

$\Rightarrow \lim_{z\to\infty} X(z)$ is finite. (What does this mean?)

Ex. 10.19) Checking the correctness of the z-transform calculation for a signal

$$X(z)=\frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \quad \text{(Example 10.3)}$$

$\lim_{z\to\infty} X(z)=1 \Rightarrow x[0]=1$: consistent $\quad x[n]=7\left(\frac{1}{3}\right)^n u[n]-6\left(\frac{1}{2}\right)^n u[n]$

10.5.10 Summary of Properties (Table 10.1, p. 775)

10.7 Analysis and Characterization of LTI Systems Using *z*-Transforms

For discrete-time LTI systems,

$$Y(z) = H(z)X(z)$$

$$H(z): \text{system function or transfer function of the system}$$

$$\Big\downarrow \quad z = e^{j\omega}$$

$$H(e^{j\omega}): \text{frequency response of the system}$$

10.7.1 Causality

For a causal system

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

A discrete-time LTI system is causal
iff the ROC is the exterior of a circle, including infinity.

A discrete-time LTI system with rational system function $H(z)$ is causal iff

(a) The ROC is the exterior of a circle outside
the outermost pole (property 8)

(b) O (numerator) $\leq$ O (denominator)

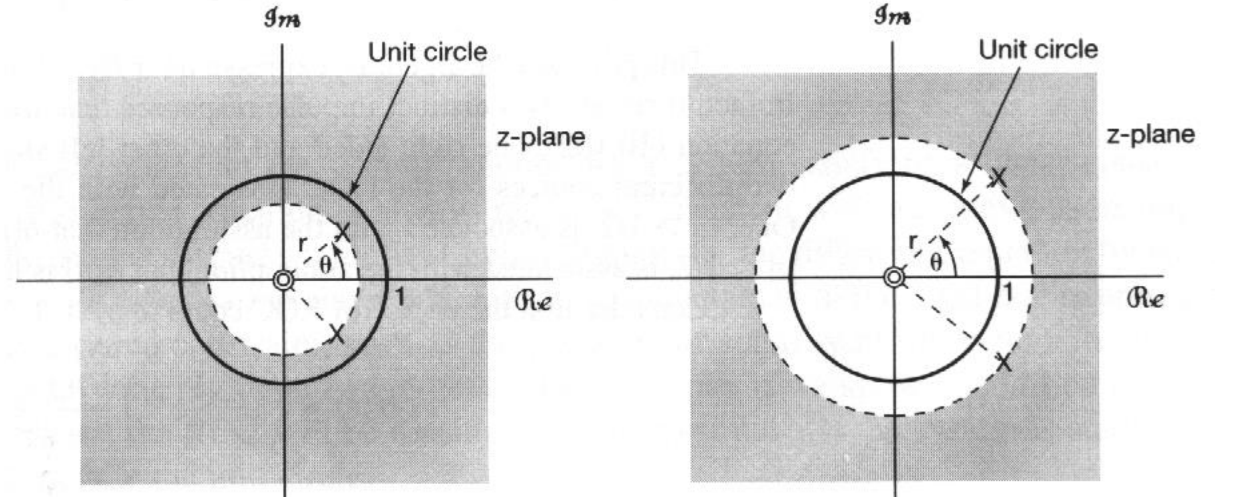
Property 8 : $X(z)$: rational, $x[n]$: right sided
=> ROC : the region in the z-plane outside the outermost pole.

10.7.2 Stability

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$ Absolutely Summable (Sufficient and Necessary)

An LTI system is stable iff the ROC of its system function *H*(*z*) includes the unit circle.

A causal LTI system with rational system function *H*(*z*) is stable iff all of the poles of *H*(*z*) lie inside the unit circle - i.e., they must all have magnitude smaller than 1.



Ex. 10.22)

$$H(z)=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1-2z^{-1}},|z|>2 \Rightarrow h[n]=\left[\left(\frac{1}{2}\right)^n+2^n\right]u[n] \text{ (causal but unstable)}$$

However, ROC: $\frac{1}{2}<|z|<2$

$$h[n]=\left(\frac{1}{2}\right)^n u[n]-2^n u[-n-1] \text{ (noncausal but stable)}$$

$$\text{ROC}:|z|<\frac{1}{2},\quad h[n]=-\left[\left(\frac{1}{2}\right)^n+2^n\right]u\left[-n-1\right] \text{ (neither causal nor stable)}$$

10.7.3 LTI Systems Characterized by Linear Constant-Coefficient Difference Equations

$$y[n]-\frac{1}{2}y[n-1]=x[n]+\frac{1}{3}x[n-1]$$

$$Y(z)-\tfrac{1}{2}z^{-1}Y(z)=X(z)+\tfrac{1}{3}z^{-1}X(z)$$

$$Y(z)=X(z)\left[\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right] \qquad H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$\text{i) ROC}: |z|>\frac{1}{2},\quad h[n]=\left(\frac{1}{2}\right)^{n}u[n]+\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

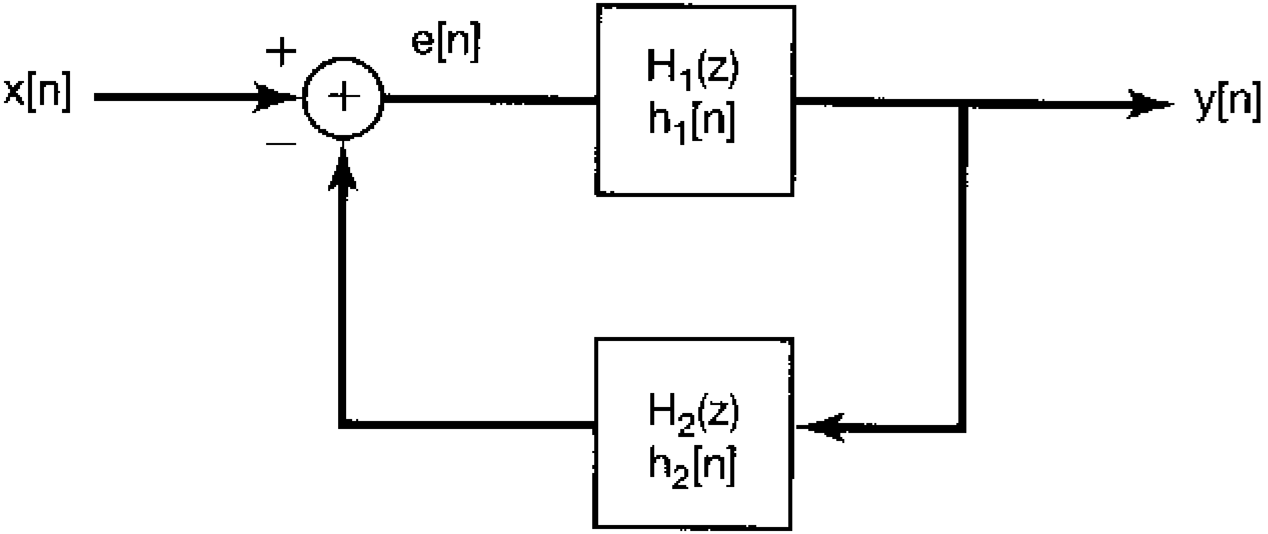
$$\text{ii) ROC}: |z|<\frac{1}{2},\quad h[n]=-\left(\frac{1}{2}\right)^{n}u[-n-1]-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^{n-1}u[-n]$$

(anticausal & unstable)

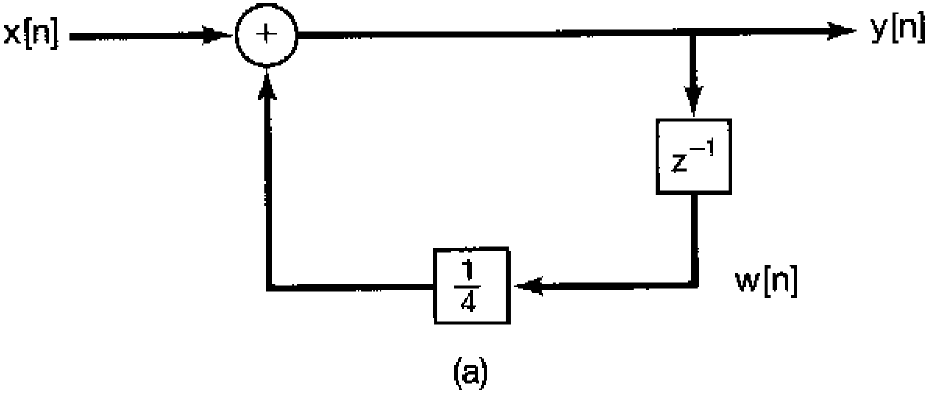
$\therefore$ We need $H(z)$ and an additional constraint of the causality or the stability.

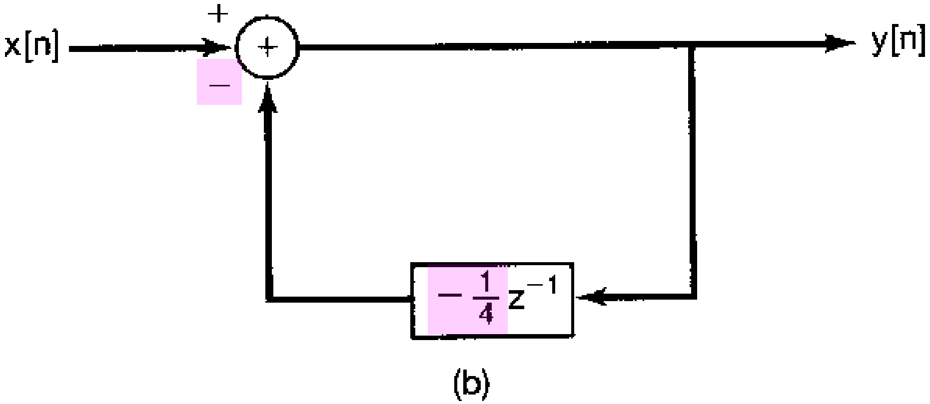
10.8 System function algebra and block diagram representations

$$\frac{Y(z)}{X(z)} = H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

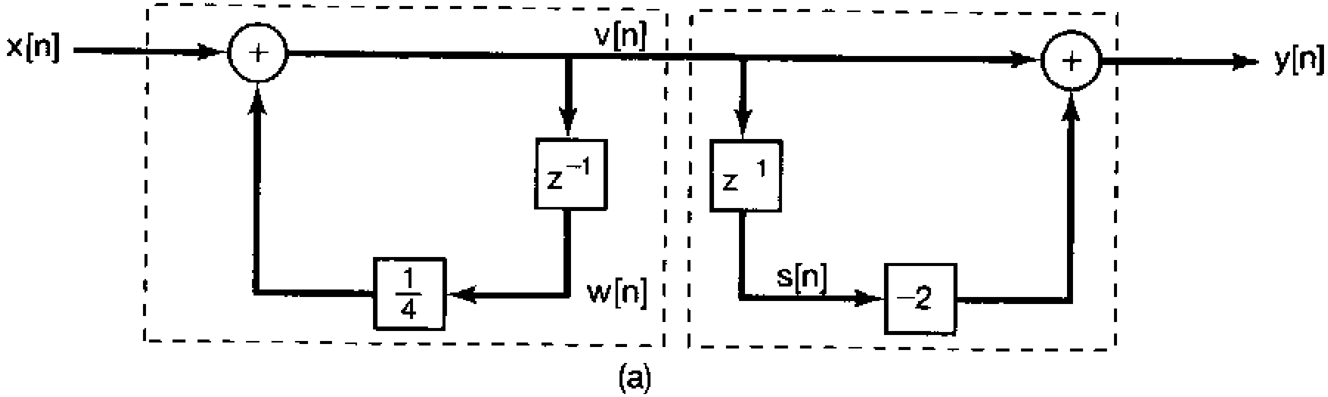


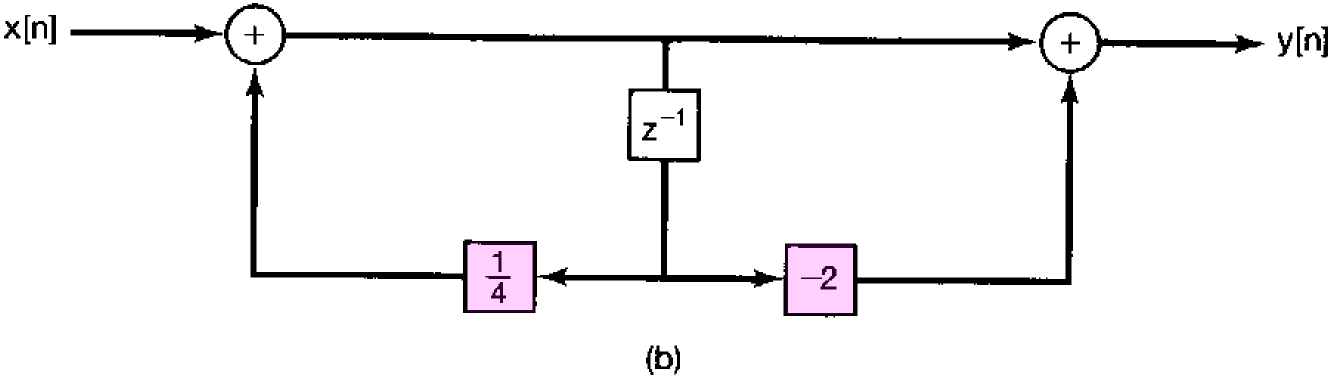
Ex. 10.28) $y[n]-\frac{1}{4}y[n-1]=x[n]$ $H(z)=\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$





Ex. 10.29) $$H(z)=\frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}=\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(1-2z^{-1}\right)$$

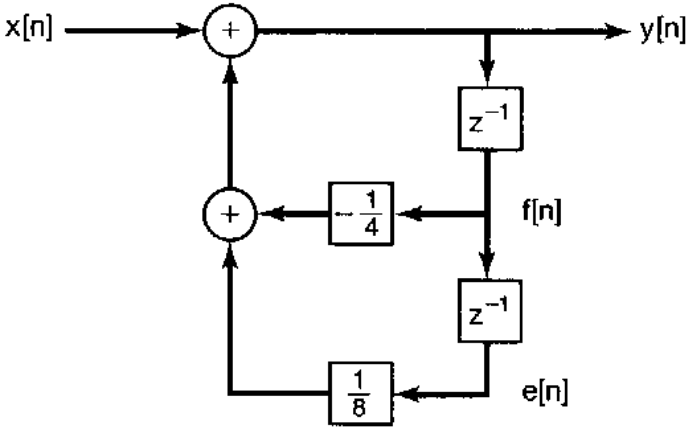




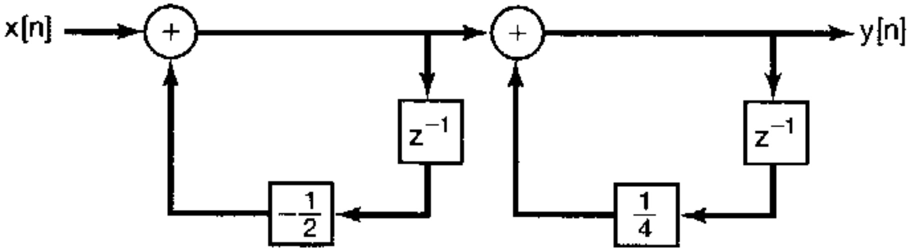
Ex. 10.30)

$$H(z)=\frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]$$



(a) direct form



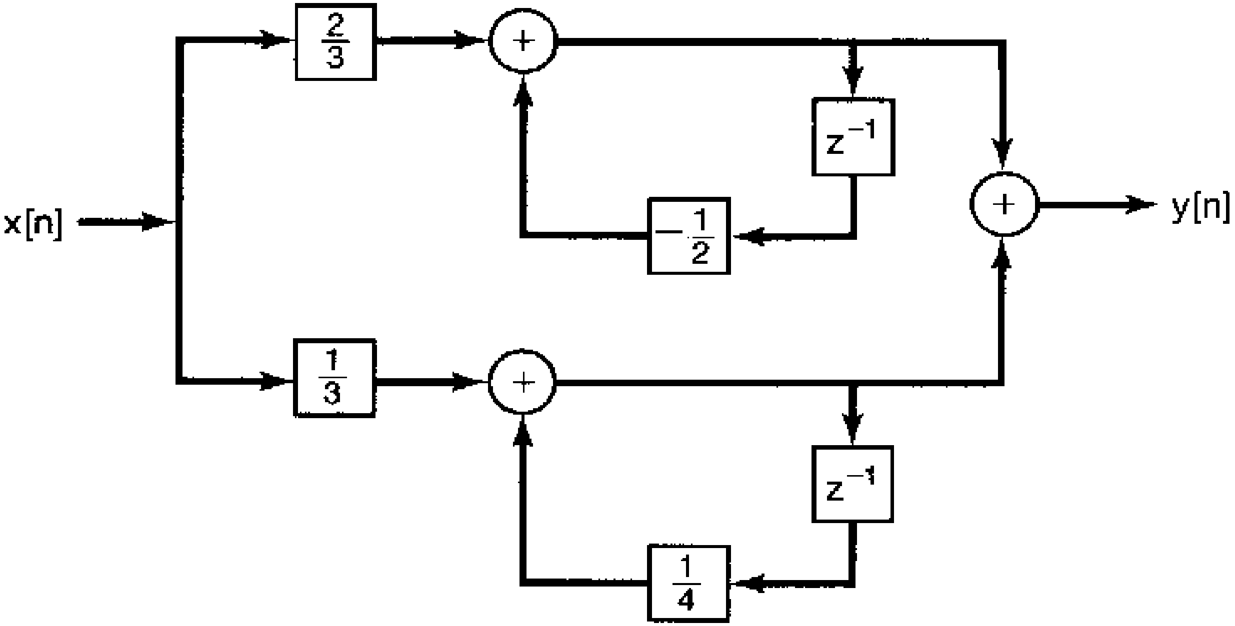
(b) cascade form

$$H(z)=\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

(c) parallel form: Partial fraction

$$H(z)=\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

$$=\frac{\frac{2}{3}}{\left(1+\frac{1}{2}z^{-1}\right)}+\frac{\frac{1}{3}}{\left(1-\frac{1}{4}z^{-1}\right)}$$



Ex. 10.31) $H(z)=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$

i) Direct-form representation

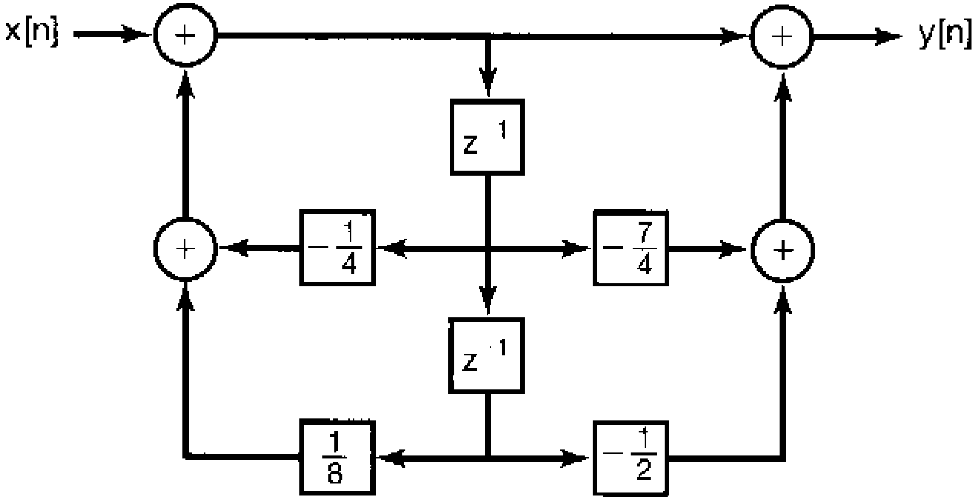
$$H(z)=\left(\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}\right)\left(1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}\right)$$

ii) Cascade-form

$$H(z)=\left(\frac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}\right)$$

iii) Parallel-form

$$H(z)=4+\frac{5/3}{1+\frac{1}{2}z^{-1}}-\frac{14/3}{1-\frac{1}{4}z^{-1}}$$



10.9 The Unilateral *z*-Transform

- Bilateral *z*-transform vs. unilateral *z*-transform
- Unilateral

  - useful in analyzing causal systems specified by linear constant-coefficient difference equations with nonzero initial conditions (i.e., not initially at rest)

  - Notation

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{UZ}} \mathcal{X}(z) = \mathcal{UZ}\{x[n]\}$$

  - the bilateral transform of $x[n]u[n]$

  - ROC : the exterior of a circle

10.9.1 Examples of Unilateral *z*-transform and Inverse Transforms

Ex. 10.33) $x[n] = a^{n+1}u[n+1]$

$$X(z) = \frac{z}{1-az^{-1}}, \quad |z| > |a|$$

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{n+1}z^{-n}$$

$$= \frac{a}{1-az^{-1}}, \quad |z| > |a|$$

Ex. 10.34) (compare Examples 10.9 ~ 10.11)

$$\mathcal{X}(z)=\frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

- ROC <u>must</u> be the exterior of the circle

$$\therefore |z|>\frac{1}{3}$$

$$\Rightarrow x[n]=\left(\frac{1}{4}\right)^n u[n]+2\left(\frac{1}{3}\right)^n u[n] \quad \text{for} \quad n\geq 0$$

- Inverse unilateral *z*-transforms

  - long division in the ROC $|z| > |a|$

$$\mathcal{X}(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

- Rational function of $z : \dfrac{p(z)}{q(z)}$

  - for this to be unilateral transform,

  Deg. (numerator) $\leq$ Deg. (denominator)

10.9.2 Properties of the Unilateral *z*-transform

- Identical to the bilateral counterparts
  - Linearity
  - Scaling in the *z*-Domain
  - Time expansion
  - Conjugation
  - Differentiation in the *z*-Domain
- Fundamentally a unilateral property
  - Initial-value theorem ($\because$ requirement : $x[n]=0$ for $n < 0$)
- No meaningful
  - Time-reversal property

- Identical in the convolution property
  - If $x_1[n] = x_2[n] = 0$ for all $n < 0$, then

$$\Rightarrow x_1[n] * x_2[n] \xleftrightarrow{\mathcal{UZ}} \mathcal{X}_1(z)\mathcal{X}_2(z)$$

Ex. 10.36) Causal LTI system

$y[n] + 3y[n-1] = x[n]$ with the condition of initial rest

$$\mathcal{H}(z) = \frac{1}{1+3z^{-1}}$$

If $x[n] = \alpha u[n]$, the unilateral (and bilateral) z-transform of $y[n]$

$$\mathcal{Y}[z] = \mathcal{H}(z)\mathcal{X}(z) = \frac{\alpha}{\left(1+3z^{-1}\right)\left(1-z^{-1}\right)} = \frac{(3/4)\alpha}{1+3z^{-1}} + \frac{(1/4)\alpha}{1-z^{-1}}$$

$$\Rightarrow y[n] = \alpha\left[\frac{1}{4} + \left(\frac{3}{4}\right)(-3)^n\right]u[n]$$

- Difference in the convolution property
  - If $x_1[n]$ or $x_2[n]$ is nonzero for $n < 0$,

$$\mathcal{Z}\{x_1[n] * x_2[n]\} = \mathcal{Z}\{x_1[n]\} \cdot \mathcal{Z}\{x_2[n]\}$$

$$\mathcal{UZ}\{x_1[n] * x_2[n]\} \neq \mathcal{UZ}\{x_1[n]\} \cdot \mathcal{UZ}\{x_2[n]\}$$

- The shifting property for the unilateral transform

i) $y[n]=x[n-1]$

$$\mathcal{Y}(z)=\sum_{n=0}^{\infty}x[n-1]z^{-n}=x[-1]+\sum_{n=1}^{\infty}x[n-1]z^{-n}$$

$$=x[-1]+\sum_{n=0}^{\infty}x[n]z^{-(n+1)}=x[-1]+z^{-1}\sum_{n=0}^{\infty}x[n]z^{-n}$$

$$=x[-1]+z^{-1}\mathcal{X}(z)\qquad\text{(the time delay property)}$$

ii) $w[n]=y[n-1]=x[n-2]$

$$\mathcal{W}(z)=x[-2]+z^{-1}\mathcal{Y}(z)=x[-2]+x[-1]z^{-1}+z^{-2}\mathcal{X}(z)$$

Then, $\mathcal{UZ}\{x[n-m]\}=?$

- Time advance property for unilateral transforms

$$x[n+1]\xleftrightarrow{\mathcal{UZ}}z\mathcal{X}(z)-zx[0]$$

pf) Problem 10.60

10.9.3 Solving Difference Equations Using the Unilateral *z*-Transform

Ex. 10.37) causal LTI system

$$y[n]+3y[n-1]=x[n]$$

$$x[n]=\alpha u[n], \qquad y[-1]=\beta$$

$$\mathcal{Y}(z)+3\beta+3z^{-1}\mathcal{Y}(z)=\frac{\alpha}{1-z^{-1}}$$

$$\mathcal{Y}(z)=\underbrace{\boxed{-\frac{3\beta}{1+3z^{-1}}}}_{\text{zero-input response}}+\underbrace{\boxed{\frac{\alpha}{\left(1+3z^{-1}\right)\left(1-z^{-1}\right)}}}_{\text{zero-state response}}$$

$$\mathcal{Y}(z)=\frac{3}{1+3z^{-1}}+\frac{2}{1-z^{-1}} \qquad (\alpha=8 \;\&\; \beta=1)$$

$$\Rightarrow y[n]=[3(-3)^{n}+2]u[n], \quad n\geq 0$$