2.3 Properties of linear time-invariant systems

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Only for LTI systems.

The characteristics of an LTI system are completely determined by its impulse response.

Example 2.9 - nonlinear system

$$h[n] = \begin{cases} 1, & n = 0, 1 \\ 0, & otherwise \end{cases} \quad \Rightarrow \quad \begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] + x[n-1] \end{aligned}$$

Nonlinear systems with the same unit impulse responses

$$y[n] = (x[n] + x[n-1])^2 \qquad y[n] = \max(x[n], x[n-1])$$

Consider for unit step input!

2.3.1 The Commutative Property

$$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{+\infty}h[k]x[n-k]$$

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{+\infty}h(\tau)x(t-\tau)d\tau$$

Proof) $x[n]*h[n]=\sum_{k=-\infty}^{+\infty}x[k]h[n-k]$

$k=n-r$

$$=\sum_{r=+\infty}^{-\infty}x[n-r]h[r]$$

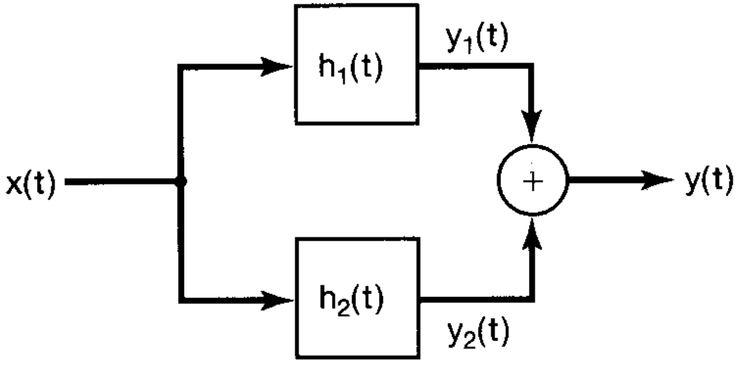
$$=\sum_{r=-\infty}^{+\infty}h[r]x[n-r]$$

$$=h[n]*x[n]$$

2.3.2 The Distributive Property

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$



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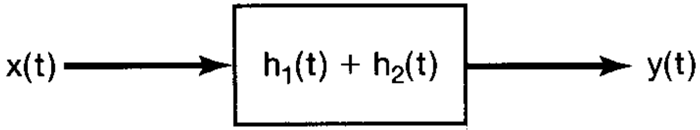
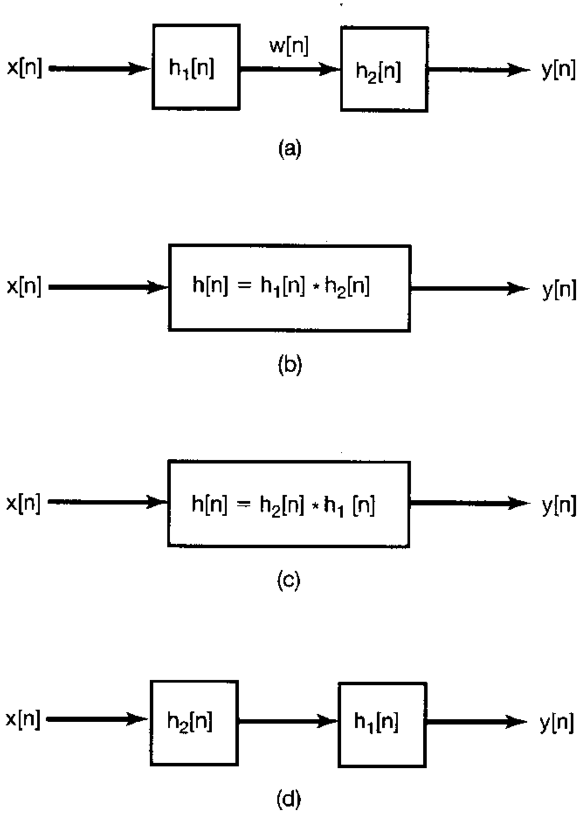


(b)

2.3.3 The Associative Property

$$x[n]*(h_1[n]*h_2[n])=(x[n]*h_1[n])*h_2[n]$$

$$x(t)*[h_1(t)*h_2(t)]=[x(t)*h_1(t)]*h_2(t)$$



2.3.4 LTI Systems with and without Memory

$$h[n] = 0 \quad \text{for} \quad n \neq 0 \qquad h(t) = 0 \quad \text{for} \quad t \neq 0$$

$$h[n] = K\delta[n] \qquad h(t) = K\delta(t)$$

$$y[n] = Kx[n] \qquad y(t) = Kx(t)$$

2.3.5 Invertibility of LTI Systems

$$(h(t) * h_1(t) = \delta(t), \quad h[n] * h_1[n] = \delta[n])$$

$$h[n] = 2\delta[n] \Leftrightarrow h_1[n] = h^{-1}[n] = \frac{1}{2}\delta[n]$$

$$h[n] = \delta[n - n_0] \Leftrightarrow h_1[n] = h^{-1}[n] = \delta[n + n_0]$$

$$h[n] = u[n] \Leftrightarrow h_1[n] = h^{-1}[n] = \delta[n] - \delta[n-1]$$

Check) $h[n] * h_1[n] = u[n] * \{\delta[n] - \delta[n-1]\}$

$$= u[n] * \delta[n] - u[n] * \delta[n-1]$$

$$= u[n] - u[n-1]$$

$$= \delta[n]$$

2.3.6 Causality for LTI Systems

$y[n]$ must not depend on $x[k]$ for $k > n$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \Rightarrow h[n-k] = 0 \quad \text{for } k > n$$

$$\Rightarrow h[n] = 0 \quad \text{for } n < 0$$

Initial rest: If the input is 0 up to some time, its output is also 0 up to that time.

$y[n] = x[n] + 3$: causal but not initial rest (not linear)

$\Rightarrow$ Only for linear systems, Causal = Initial Rest

For causal LTI systems

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

2.3.7 Stability for LTI systems

Stability : BIBO stable

$$|x[n]| < B \text{ for all } n \Rightarrow |y[n]| < B' \text{ for all } n$$

$$|y[n]| = \left| \sum_{k=-\infty}^{+\infty} h[k]x[n-k] \right|$$

$$|y[n]| \le \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|$$

$$|y[n]| \le B \sum_{k=-\infty}^{+\infty} |h[k]| \quad \text{for all } n$$

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$

Absolutely summable
(Sufficient and <u>Necessary</u>)
(Prob. 2.49)

Continuous-time system

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty$$

Absolutely integrable
(Sufficient and Necessary)

2.3.8 The Unit Step Response of an LTI System

$h[k]$ or $h(t)$: determines the behavior of an LTI system

- Unit step response, $s[n]$ or $s(t)$

$\Rightarrow$ the output when $x[n] = u[n]$ or $x(t) = u(t)$

- Unit impulse response vs. Unit step response
  - Discrete time

$$
\begin{aligned}
s[n] &= u[n] * h[n] \\
&= h[n] * u[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \\
&= \sum_{k=-\infty}^{n} h[k] \quad : \text{running sum of its impulse response}
\end{aligned}
$$

$$\therefore \; h[n] = s[n] - s[n-1]$$

2.3.8 The Unit Step Response of an LTI System

- Continuous time

$$s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{t} h(\tau) d\tau$$

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

$\Rightarrow$ In both continuous and discrete time, the unit step response can also be used to characterize an LTI system.

2.4 Causal LTI systems described by differential and difference equations

2.4.1 Linear constant-coefficient differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (\text{order} : N)$$

- Implicit specification of the system
- Find a solution → explicit expression

  need one or more auxiliary conditions.
- $y(t) = y_p(t) + y_h(t)$

  particular solution (forced response) — $y_p(t)$

  homogenous solution (natural response) — $y_h(t)$

Example 2.14 – Linear Constant-Coefficient Differential Equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad \text{when } x(t) = Ke^{3t}u(t)$$

i) particular solution (forced response)

Let $y_p(t) = Ye^{3t}$ for $t > 0$, then

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}$$

$$3Y + 2Y = K \Rightarrow Y = \frac{K}{5}$$

$$\therefore \quad y_p(t) = \frac{K}{5}e^{3t}, \quad t > 0$$

ii) homogenous solution (natural response)

$$y_h(t) = Ae^{st}$$

$$Ase^{st} + 2Ae^{st} = Ae^{st}(s+2) = 0 \Rightarrow s = -2$$

$$\therefore \quad y_h(t) = Ae^{-2t}$$

$$\therefore \quad y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \quad t > 0$$

Example 2.14 – Linear Constant-Coefficient Differential Equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad \text{when } x(t) = Ke^{3t}u(t)$$

$$\therefore \quad y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \ t > 0$$

At the condition of initial rest, $y(t) = 0, \ t < 0$

$$0 = A + \frac{K}{5} \qquad \Rightarrow A = -\frac{K}{5}$$

$$\therefore \quad y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right], \ t > 0$$

$$\Rightarrow \quad y(t) = \frac{K}{5}\left[e^{3t} - e^{-2t}\right]u(t)$$

- The condition of initial rest : if $x(t) = 0$ for $t \le t_0 \Rightarrow y(t) = 0$ for $t \le t_0$

  cf) zero initial condition at a fixed point in time.

2.4 Causal LTI systems described by differential and difference equations

2.4.2 Linear constant-coefficient difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Similar to the differential equations
- Alternative approach : recursive equation

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

- $N = 0$; nonrecursive equation $\Longrightarrow$ FIR (finite impulse response) system

$$y[n] = \sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right) x[n-k] \Longrightarrow h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- $N \ge 1$ $\Longrightarrow$ IIR (infinite impulse response) system

Example 2.15 – Linear Constant-Coefficient Difference Equation

$$y[n]-\frac{1}{2}y[n-1]=x[n] \quad \Rightarrow \quad y[n]=x[n]+\frac{1}{2}y[n-1]$$

$$\left(\begin{array}{l}\text{Suppose that we impose the condition of initial rest}\\ \text{and consider the input } \; x[n]=K\delta[n]\end{array}\right) \Rightarrow y[-1]=0$$

$$y[0]=x[0]+\frac{1}{2}y[-1]=K$$

$$y[1]=x[1]+\frac{1}{2}y[0]=\frac{1}{2}K$$

$$y[2]=x[2]+\frac{1}{2}y[1]=\left(\frac{1}{2}\right)^2 K$$

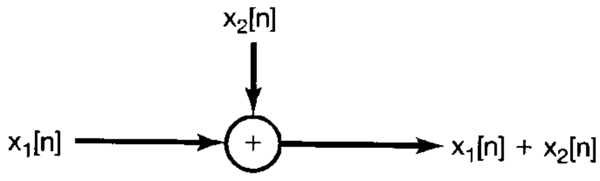
$$\vdots$$

$$y[n]=x[n]+\frac{1}{2}y[n-1]=\left(\frac{1}{2}\right)^n K$$

$$\therefore \quad h[n]=\left(\frac{1}{2}\right)^n u[n] \;\Rightarrow \text{IIR system}$$

2.4 Causal LTI systems described by differential and difference equations

2.4.3 Block diagrams



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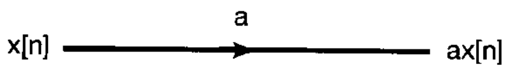

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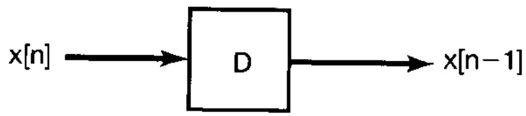

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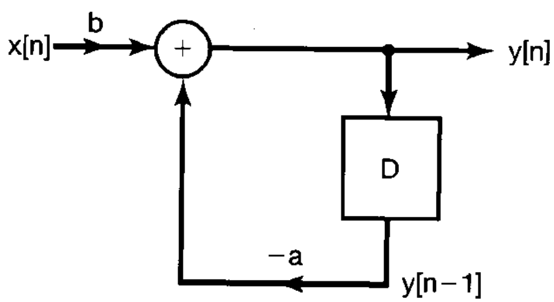
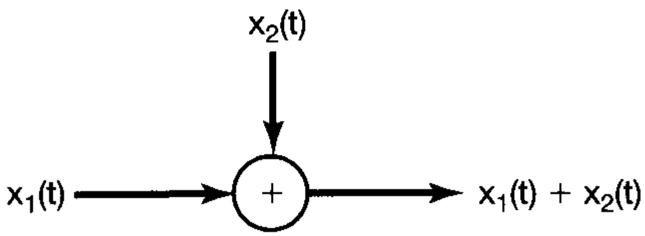


$$y[n] = bx[n] - ay[n-1]$$

$$\Rightarrow y[n] + ay[n-1] = x[n]$$



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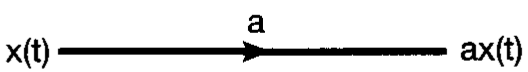

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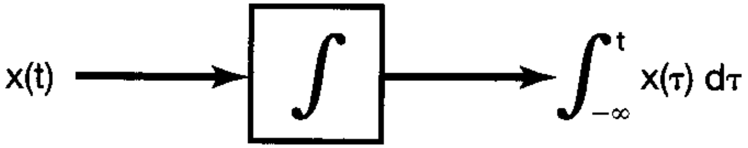
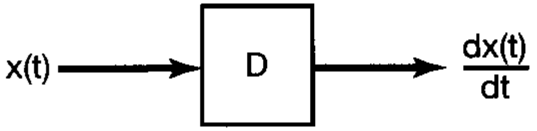
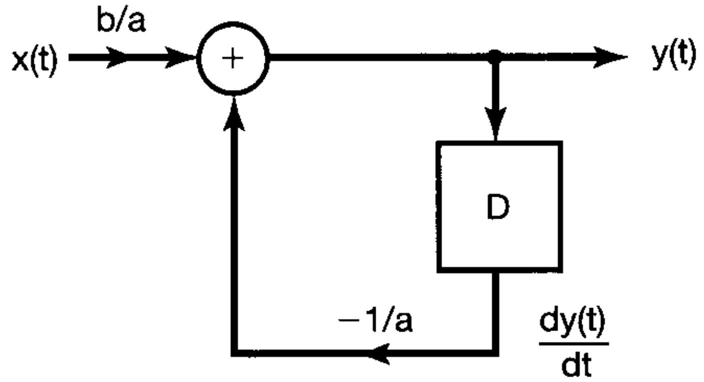




(c)

$$y(t)=\frac{b}{a}x(t)-\frac{1}{a}\frac{dy(t)}{dt}\Rightarrow\frac{dy(t)}{dt}=bx(t)-ay(t)$$



$$y(t)=\int_{-\infty}^{t}(bx(\tau)-ay(\tau))d\tau$$