4.3 Properties of the continuous-time Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

4.3.1 Linearity

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

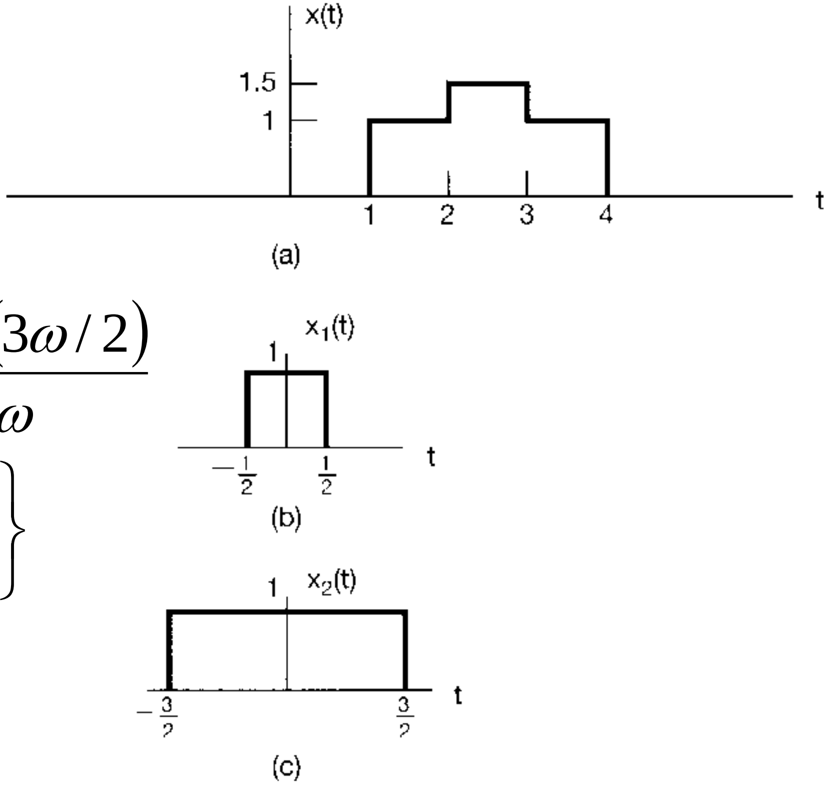
$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

$$ax(t)+by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega)+bY(j\omega)$$

4.3.2 Time shifting

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Ex. 4.9)



$$x(t) = \frac{1}{2} x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}, \quad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}$$

4.3.3 Conjugation and conjugate symmetry

$$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$$

$$\boxed{X(-j\omega) = X^*(j\omega) \quad [x(t): \ real]}$$; conjugate symmetry — ★

pf) $X^*(-j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt \right]^* = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(j\omega)$

- For $x(t)$ real & even, $\quad -t \to t$

$$X(-j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = \int_{\infty}^{-\infty} x(-t)e^{-j\omega t}(-dt) = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(j\omega)$$ — ★ ★

$$X^*(j\omega) = X(-j\omega) = X(j\omega)$$

$$\Rightarrow X(j\omega) = X^*(j\omega) \quad : \text{real function}$$

- For $x(t)$ real & odd, $\Rightarrow X(j\omega) = -X^*(j\omega) \quad$: purely imaginary function

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) : x(t) \text{ is real}$$

$$x(t) = x_e(t) + x_o(t)$$

$$\mathcal{F}\{x(t)\} = \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\}$$

$$\mathcal{E}v\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(j\omega)\}$$

$$\mathcal{O}d\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m\{X(j\omega)\}$$

4.3.4 Differentiation and integration

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \quad \left(\because \quad \frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} j\omega X(j\omega) e^{j\omega t} d\omega \right)$$

$$\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)$$

Ex. 4.11) Fourier Transform of the Unit step function

$$x(t) = u(t)$$

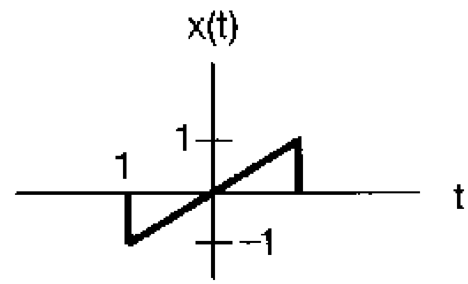
$$g(t) = \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} G(j\omega) = \quad 1$$

$$x(t) = \int_{-\infty}^{t} g(\tau)\,d\tau$$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Ex. 4.12)



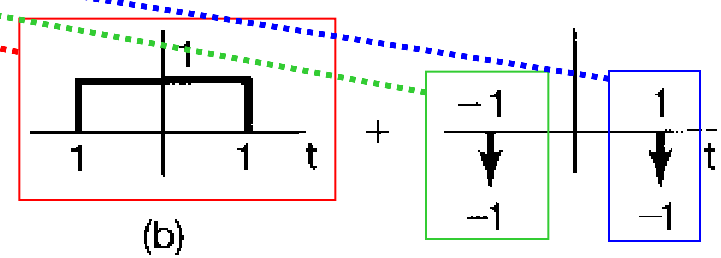
(a)

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

$$= \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

$$g(t) = \frac{dx(t)}{dt} =$$



(b)

4.3.5 Time and frequency scaling

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

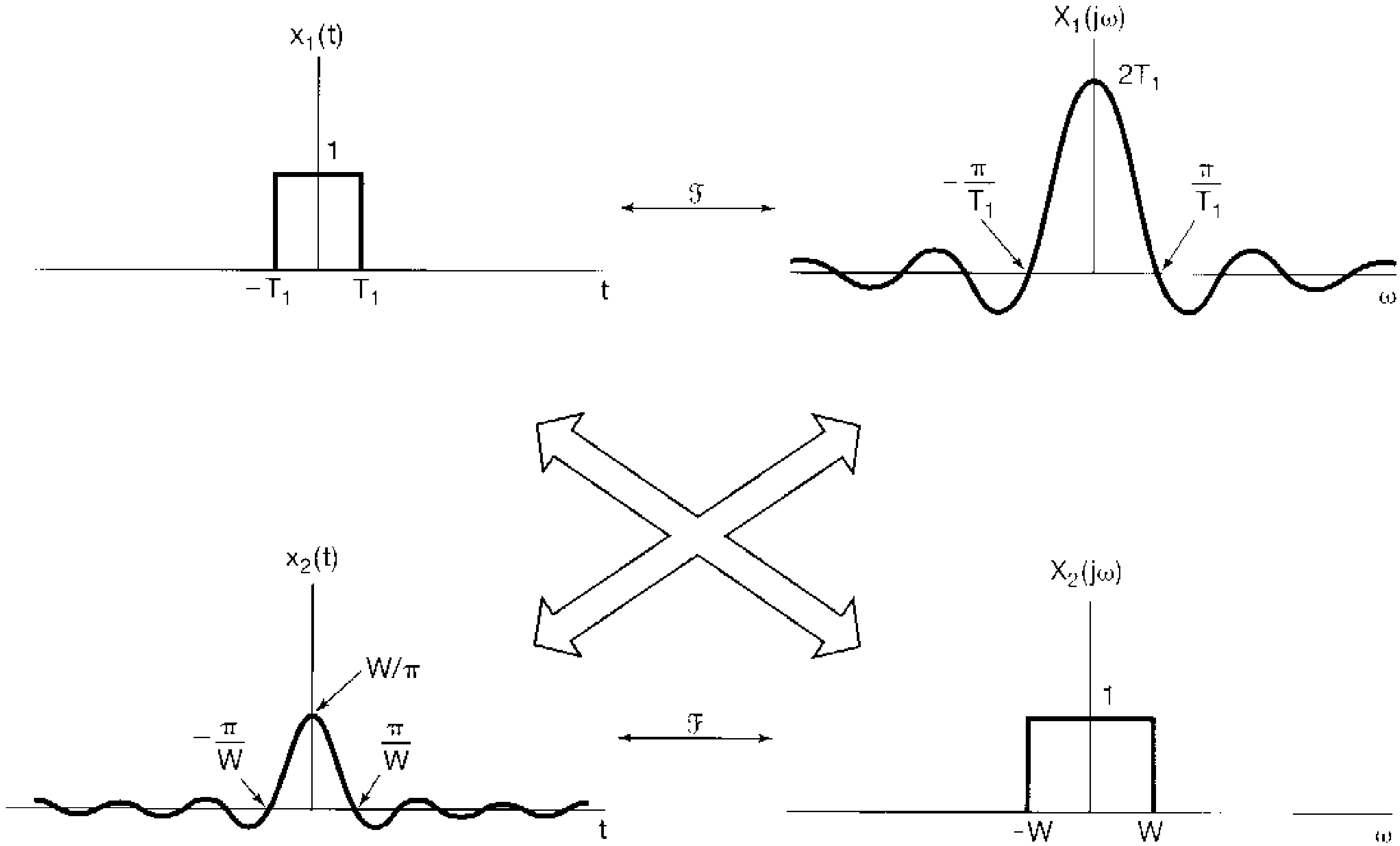
4.3.6 Duality

$$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$$

$$x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = X_1(t) = \frac{2\pi \sin tW}{\pi t} = 2\pi \frac{\sin Wt}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x_1(-\omega)$$

$$\therefore \quad \frac{\sin Wt}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} x_1(-\omega) = X_2(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$


x1(t)
1
−T1
T1
t
ℱ
X1(jω)
2T1
− π/T1
π/T1
ω
x2(t)
W/π
− π/W
π/W
t
ℱ
X2(jω)
1
-W
W
ω

Ex. 4.13) $g(t) = \dfrac{1}{1+t^2}$

$$x(t) = e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}$$

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega$$

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega \quad (t \to -t)$$

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt \quad (\omega \leftrightarrow t)$$

$$\Rightarrow \quad \mathcal{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}$$

- Other Dualities

$$\frac{dx(t)}{dt} \stackrel{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \Leftrightarrow -jtx(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$$

Proof) $\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} -jtx(t)e^{-j\omega t}dt \Leftrightarrow -jtx(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$

$$x(t-t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(jw) \Leftrightarrow e^{j\omega_0 t} x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))$$

$$\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

$$\Leftrightarrow -\frac{1}{jt} x(t) + \pi x(0)\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} X(\eta)d\eta$$

4.3.7 Parseval's relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) x^*(t) dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt$$

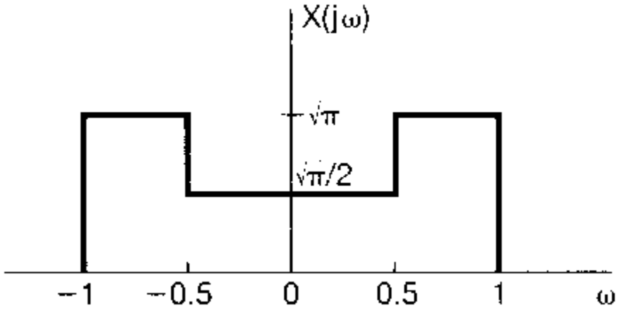
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

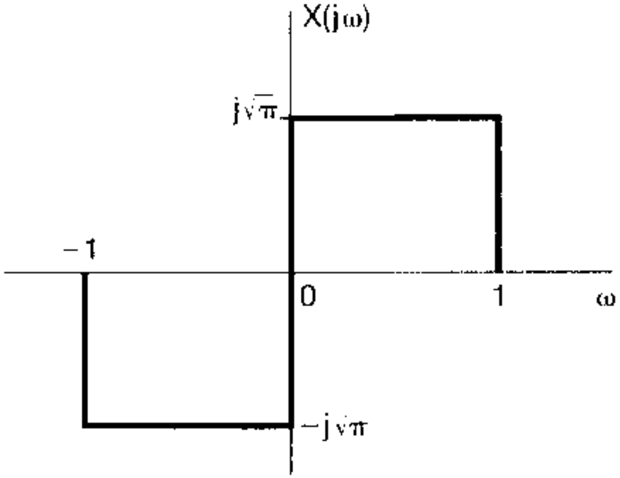
$|X(j\omega)|^2$; energy - density spectrum, $\dfrac{|X(j\omega)|^2}{2\pi}$; enegry per unit frequency

Ex. 4.14) Evaluate the following time-domain expressions: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$D = \frac{d}{dt} x(t) \Big|_{t=0}$$



(a)



(b)

Soln.)

$$E_a = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{5}{8}, \quad E_b = 1$$

$$g(t) = \frac{d}{dt} x(t) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega) = G(j\omega)$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

$$D = g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) d\omega$$

$$D_a = 0, \qquad D_b = -\frac{1}{2\sqrt{\pi}}$$

4.4 The convolution property

- Eigenfunction approach

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$H(jk\omega_0)=\int_{-\infty}^{\infty}h(t)e^{-jk\omega_0 t}dt$$

$$y(t)=\lim_{\omega_0\to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)H(j\omega)e^{j\omega t}d\omega$$

$$\Rightarrow Y(j\omega)=H(j\omega)X(j\omega)$$

- Another approach (time-domain convolution)

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$Y(j\omega) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau$$

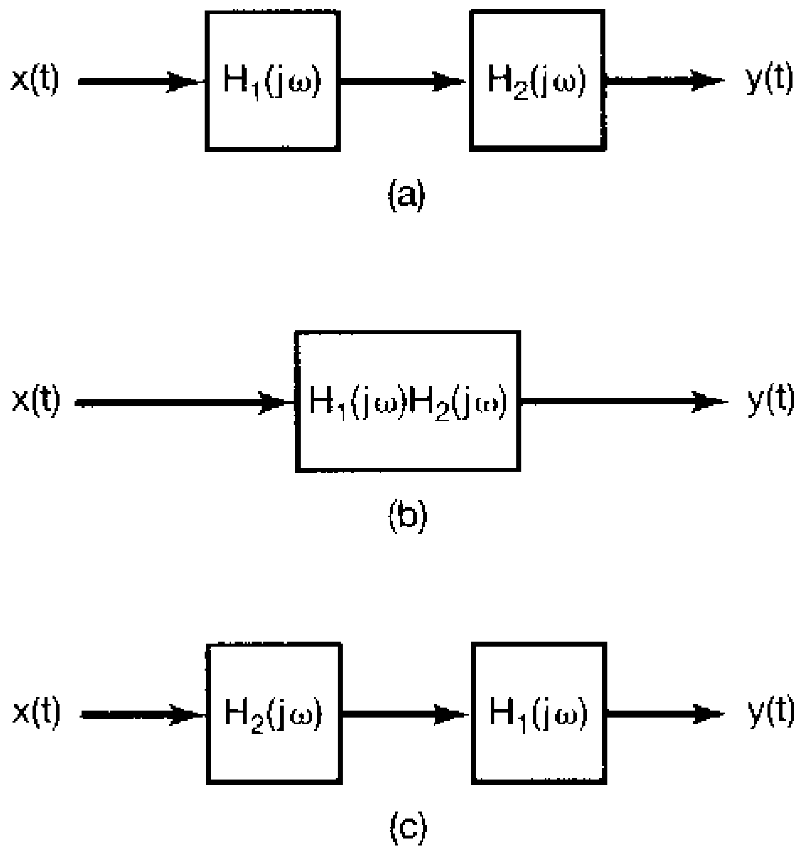
$$= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega(t-\tau)}dt\right]d\tau$$

$$\Downarrow$$

$$H(j\omega)$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$y(t) = h(t) * x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$



$$H(j\omega) = H_1(j\omega)H_2(j\omega)$$

$$h(t) = h_1(t) * h_2(t)$$

Ex. 4.15)

$$h(t) = \delta(t - t_o)$$

$$H(j\omega) = e^{-j\omega t_o}$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= e^{-j\omega t_o} X(j\omega)$$

Ex. 4.19) $h(t) = e^{-at}u(t) \quad , \quad a > 0$

$x(t) = e^{-bt}u(t) \quad , \quad b > 0$

$y(t) = x(t) * h(t)$

$$X(j\omega) = \frac{1}{b + j\omega} \quad , \quad H(j\omega) = \frac{1}{a + j\omega}$$

$$\Rightarrow Y(j\omega) = \frac{1}{(b + j\omega)(a + j\omega)}$$

using partial fraction

If $a \neq b$

$$Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right] \Rightarrow y(t) = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]u(t)$$

If $a = b$

$$Y(j\omega) = \frac{1}{(a + j\omega)^2} \Rightarrow y(t) = te^{-at}u(t)$$

Ex. 4.20)

$$x(t) = \frac{\sin \omega_i t}{\pi t} \quad , \quad h(t) = \frac{\sin \omega_c t}{\pi t} \ \text{(LPF)}$$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

$$Y(j\omega) = \begin{cases} 1 & |\omega| \le \min[\omega_i, \omega_c] \\ 0 & elsewhere \end{cases}$$

$$y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t} & \omega_c \le \omega_i \\ \dfrac{\sin \omega_i t}{\pi t} & \omega_i \le \omega_c \end{cases}$$

4.5 The multiplication property

Convolution property in the frequency domain (duality):

$$r(t) = s(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

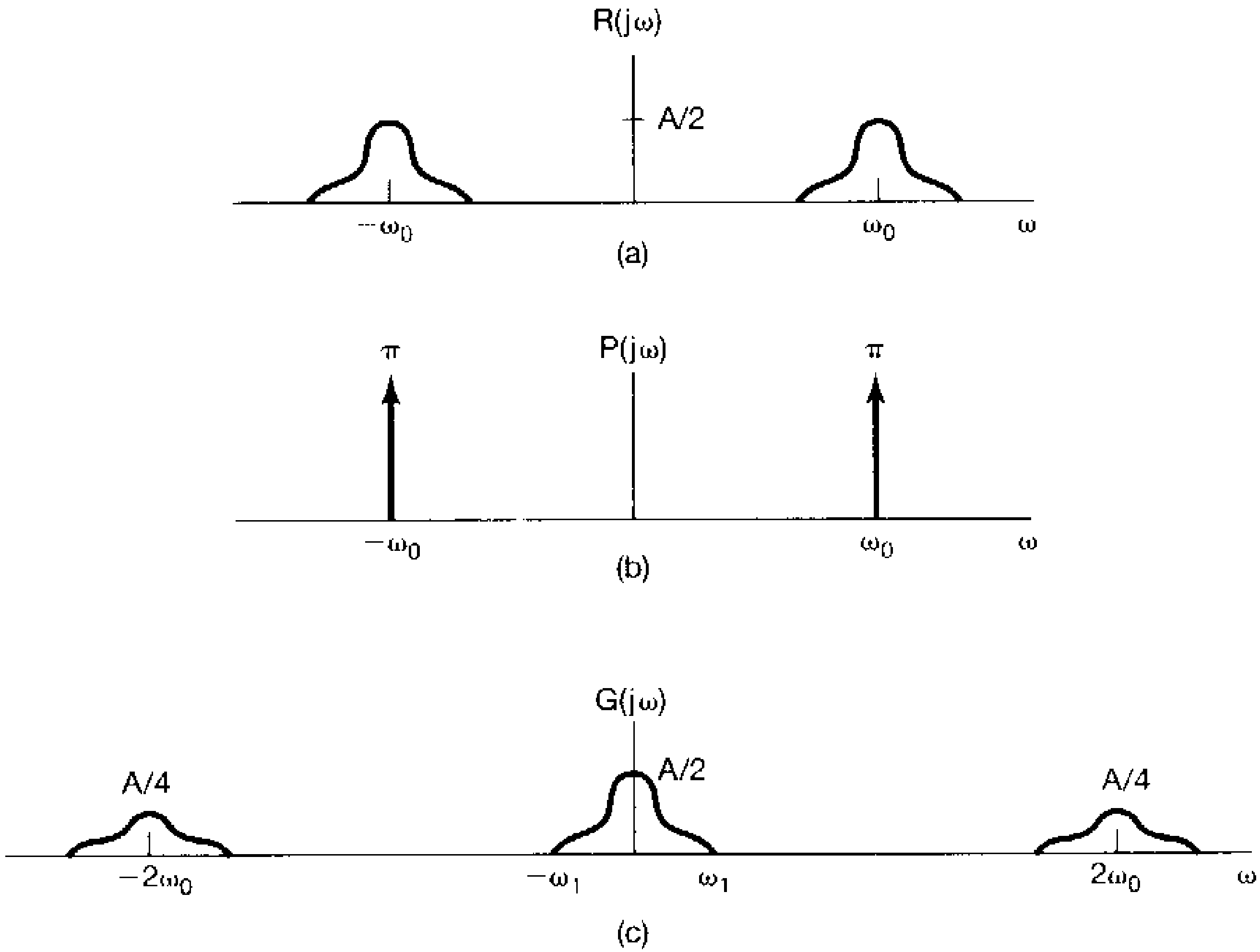
Ex. 4.21)

$$p(t) = \cos \omega_0 t$$

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$p(t)s(t) \overset{\mathcal{F}}{\longleftrightarrow} 0.5S(\omega - \omega_0) + 0.5S(\omega + \omega_0)$$

Ex. 4.22) $g(t) = r(t)p(t) \overset{\mathcal{F}}{\longleftrightarrow} G(j\omega) = \frac{1}{2\pi}[R(j\omega) * P(j\omega)]$

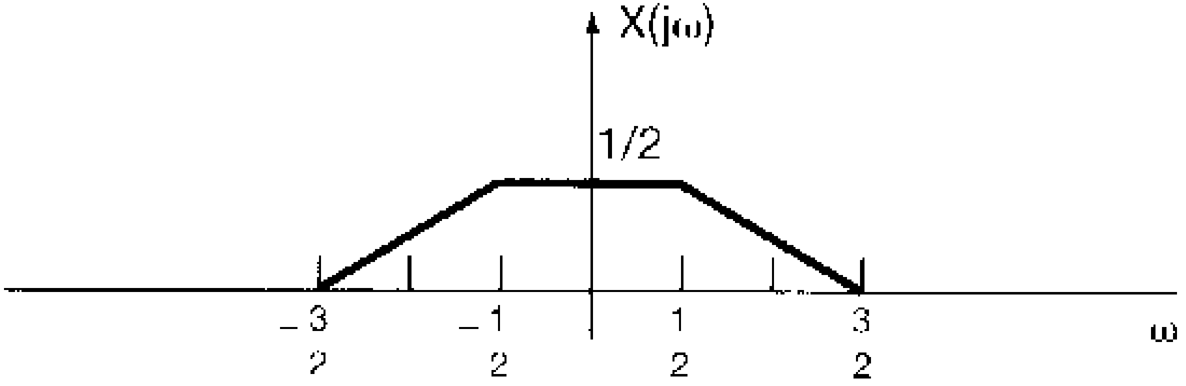


Ex. 4.23)

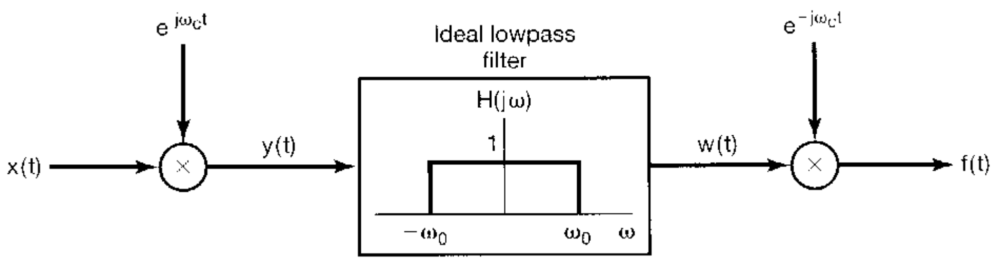
$$x(t) = \frac{\sin t \cdot \sin \frac{t}{2}}{\pi t^2}$$

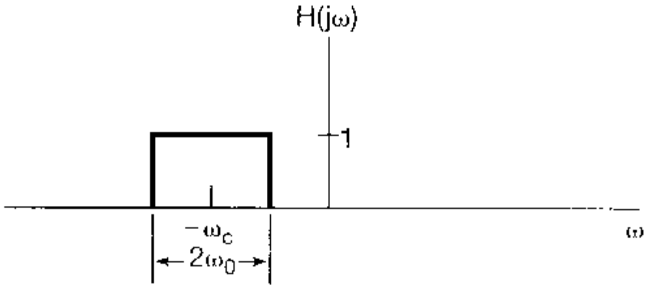
$$x(t) = \pi \left(\frac{\sin t}{\pi t} \right) \left(\frac{\sin \frac{t}{2}}{\pi t} \right)$$

$$X(j\omega) = \frac{1}{2} \mathcal{F}\left\{ \frac{\sin t}{\pi t} \right\} * \mathcal{F}\left\{ \frac{\sin \frac{t}{2}}{\pi t} \right\}$$

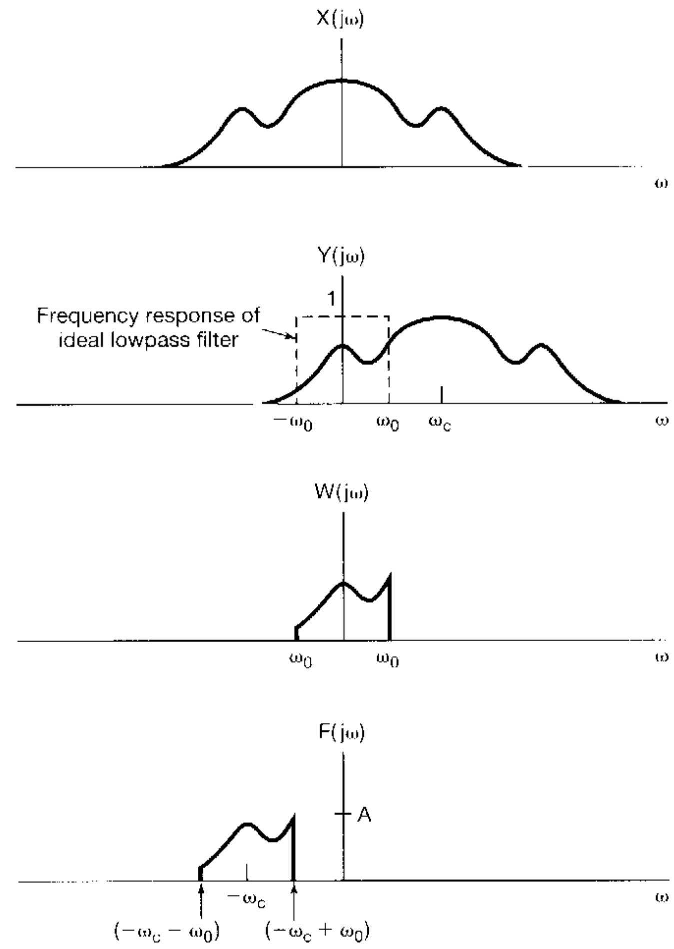


4.5.1 Frequency-selective filtering with variable center frequency





Bandpass filter



4.7 Systems characterized by linear constant-coefficient differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

taking Fourier transform

Assumption

$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$ converges

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Ex. 4.25) Consider a stable LTI system

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$H(j\omega)=\frac{(j\omega)+2}{(j\omega)^2+4(j\omega)+3}$$

$$=\frac{j\omega+2}{(j\omega+1)(j\omega+3)}$$

$$=\frac{\frac{1}{2}}{j\omega+1}+\frac{\frac{1}{2}}{j\omega+3}$$

$$\therefore \quad h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

Ex. 4.26) Consider a stable LTI system

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t) \quad \text{and} \quad x(t)=e^{-t}u(t)$$

$$Y(j\omega)=H(j\omega)X(j\omega)=\left[\frac{j\omega+2}{(j\omega+1)(j\omega+3)}\right]\left[\frac{1}{j\omega+1}\right]=\frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}$$

$$=\frac{A_{11}}{j\omega+1}+\frac{A_{12}}{(j\omega+1)^2}+\frac{A_{21}}{j\omega+3}$$

$$=\frac{\frac{1}{4}}{j\omega+1}+\frac{\frac{1}{2}}{(j\omega+1)^2}-\frac{\frac{1}{4}}{j\omega+3}$$

$$\therefore \quad y(t)=\left[\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{4}e^{-3t}\right]u(t)$$