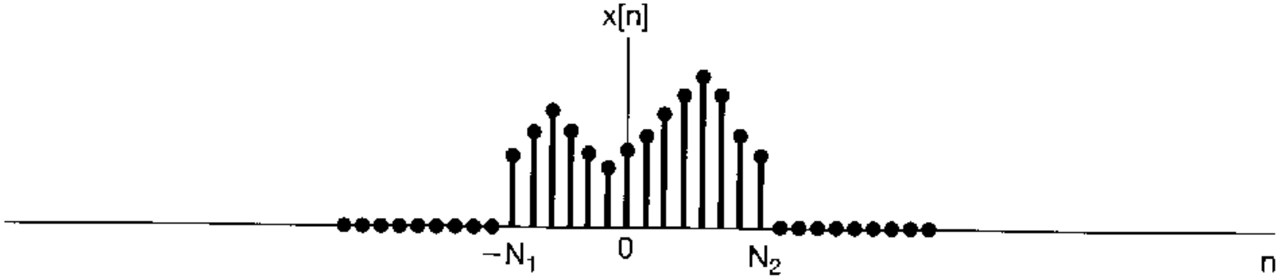
5

The Discrete-Time Fourier Transform

5.1 Representation of Aperiodic Signals
: The Discrete-time Fourier Transform

5.1.1 Development of the Discrete-Time Fourier Transform



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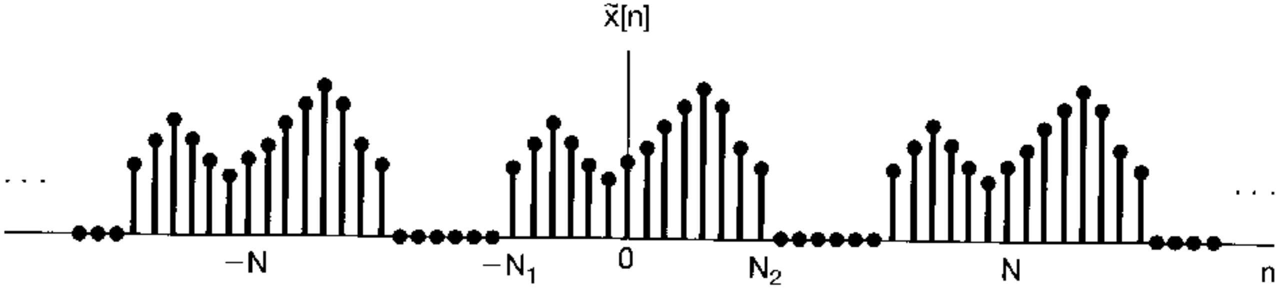


(b)

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

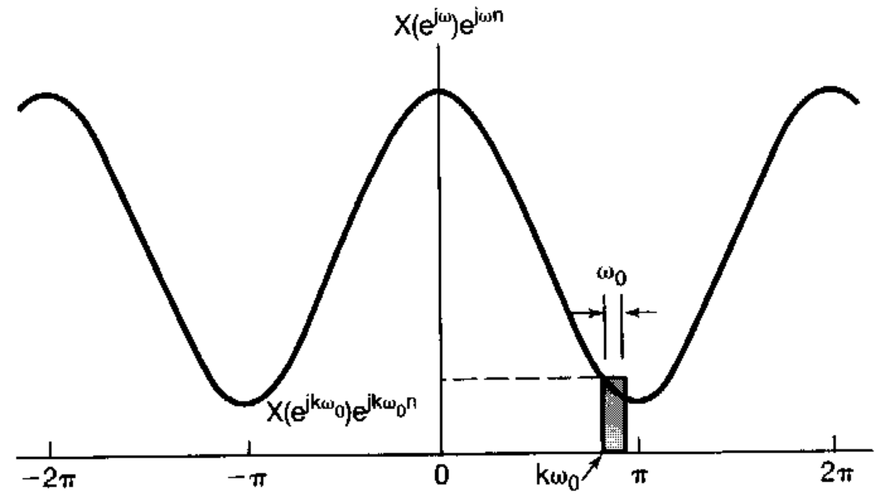
$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

Let us define $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$a_k = \frac{1}{N} X(e^{jk\omega_0}) \qquad \omega_0 = 2\pi / N$$

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}$$

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$



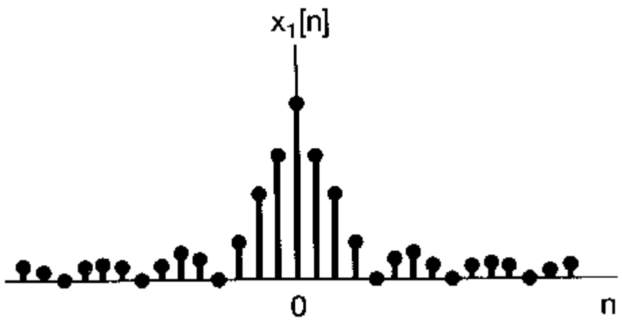
As N goes to infinity, ω_0 becomes 0, $k\omega_0 \to \omega$, and $\tilde{x}[n] \to x[n]$.

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$: Synthesis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$: Analysis equation

$X(e^{j\omega})$: discrete - time Fourier Transform(DTFT)

: the spectrum of $x[n]$



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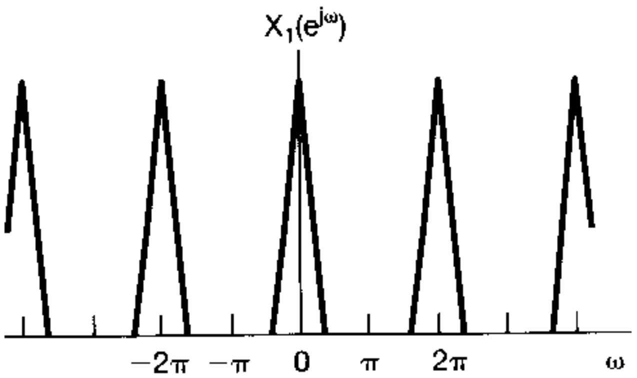

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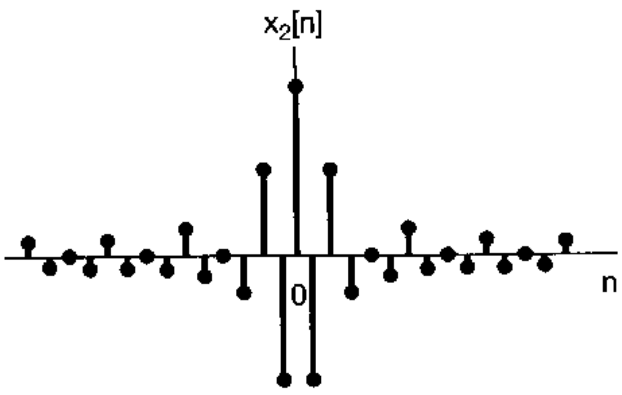

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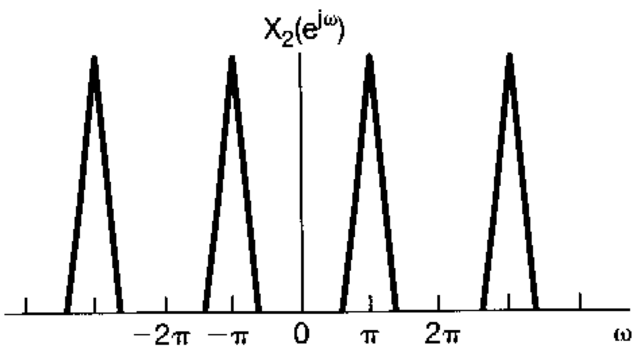
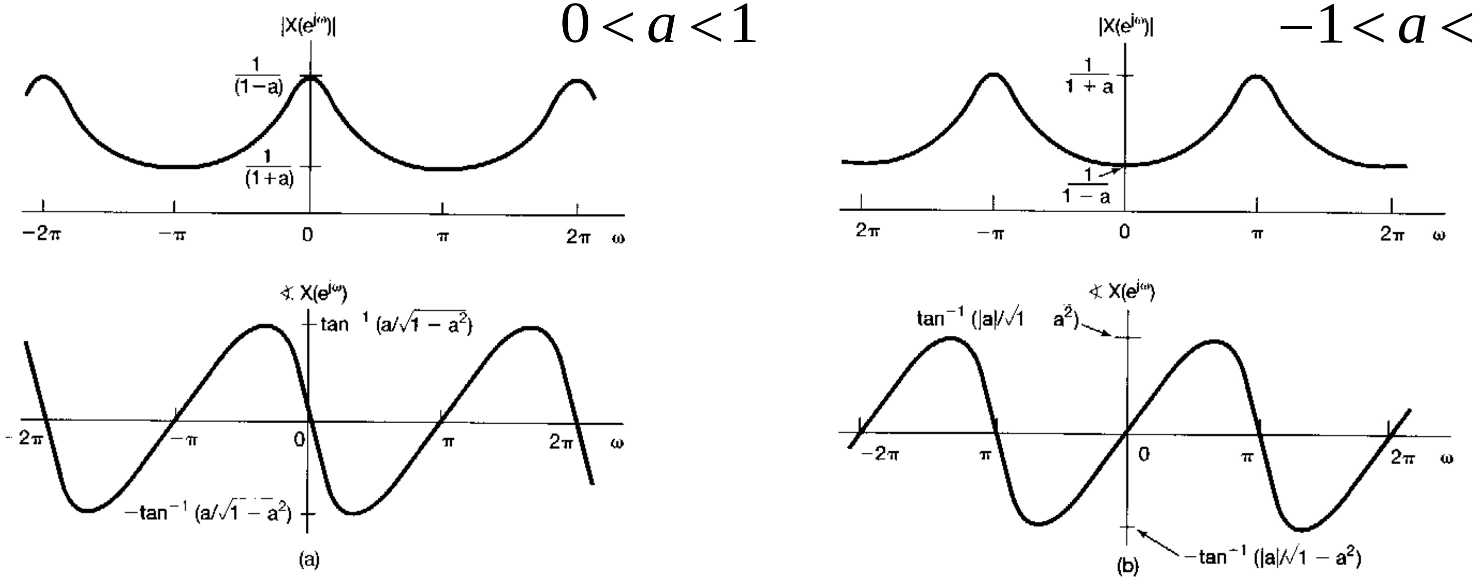


(d)

5.1.2 Examples of Discrete-Time Fourier Transforms

Ex. 5.1)

$0 < a < 1$ $\quad\quad$ $-1 < a < 0$



$$x[n] = a^n u[n], \qquad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

Ex. 5.2)

$$x[n] = a^{|n|}, \qquad |a| < 1$$

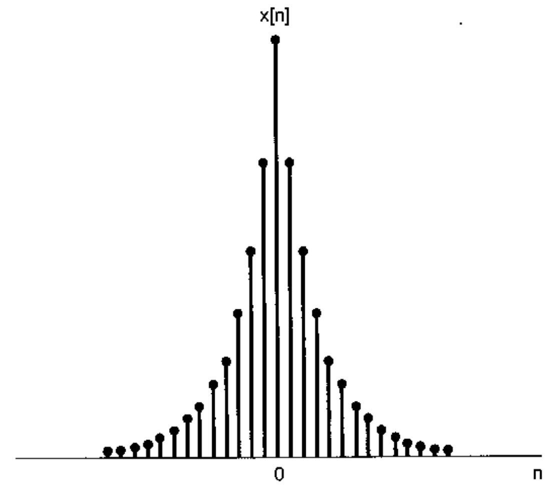
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

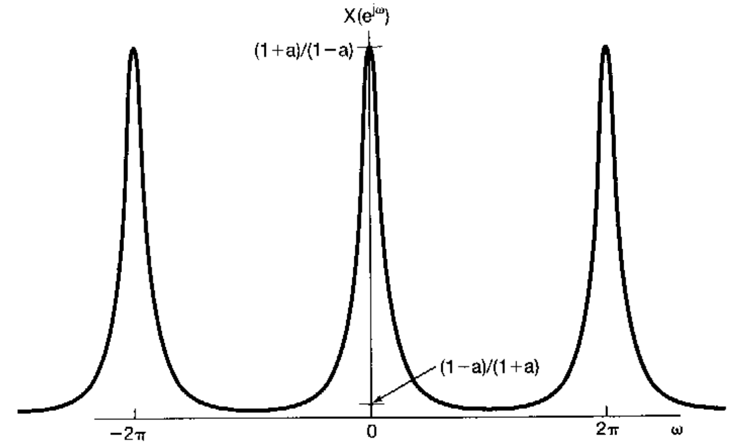
$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{m=1}^{\infty} (ae^{j\omega})^{m}$$

$$= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{1-a^{2}}{1-2a\cos\omega + a^{2}}$$



(a)

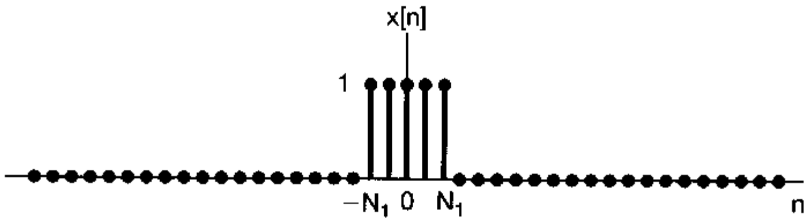


(b)

$0 < a < 1$

Ex. 5.3) $x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases}$

$$X(e^{j\omega})=\sum_{n=-N_1}^{N_1}e^{-j\omega n}=\frac{\sin\omega\left(N_1+\frac{1}{2}\right)}{\sin(\omega/2)}$$



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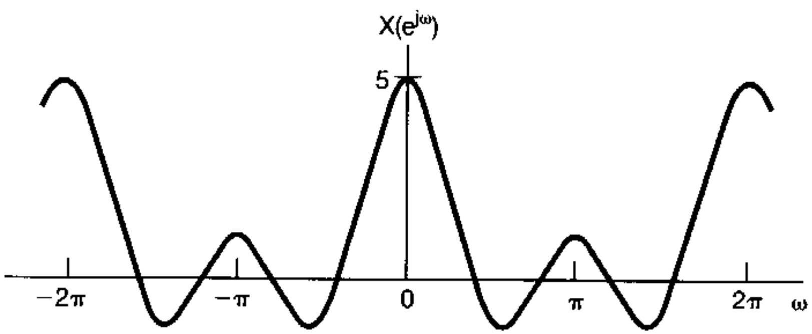
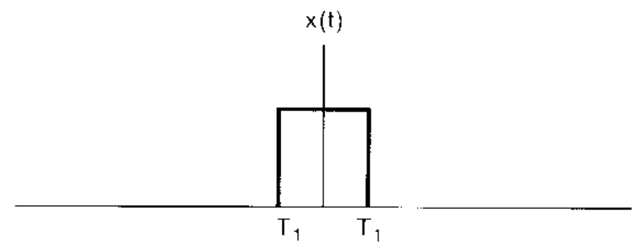


(b)

Note) Continuous-Time Fourier Transform

Ex. 4.4) $x(t)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}$

$$X(j\omega)=\int_{-T_1}^{+T_1}e^{-j\omega t}dt=2\frac{\sin\omega T_1}{\omega}$$



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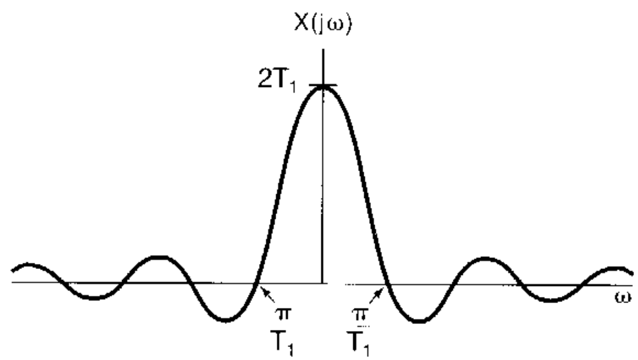


(b)

5.1.3 Convergence issues associated with the discrete-time Fourier transform

- The analysis equation

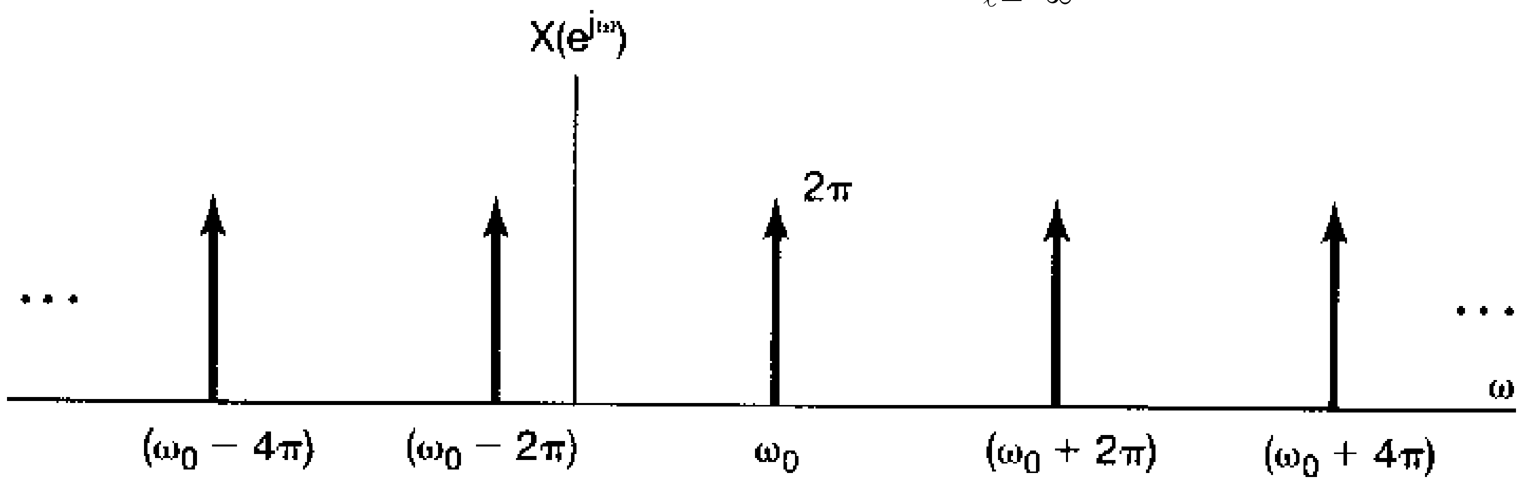
$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty \qquad \text{or} \qquad \sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

- The synthesis equation

: no issues of convergence

5.2 The Fourier transform for periodic signals

$$x[n] = e^{j\omega_0 n} \qquad X(e^{j\omega}) = \sum_{\ell=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi\ell)$$



Check) $$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{\ell=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi\ell)e^{j\omega n}d\omega$$

$$= e^{j\omega_0 n}$$

$$x[n] = e^{jk\omega_0 n} \Longrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - k\omega_0 - 2\pi l)$$

$$x[n] = a_k e^{jk\omega_0 n} \Longrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0 - 2\pi l)$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \Longrightarrow X(e^{j\omega}) = \sum_{k=<N>} \sum_{l=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0 - 2\pi l)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{k=<N>} 2\pi a_k \delta(\omega - k\omega_0 - 2\pi l)$$

$$= \sum_{l=-\infty}^{\infty} \sum_{k=<N>} 2\pi a_k \delta(\omega - \frac{2\pi}{N}(k + Nl))$$

$$= \sum_{m=-\infty}^{\infty} 2\pi a_m \delta(\omega - \frac{2\pi}{N} m)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

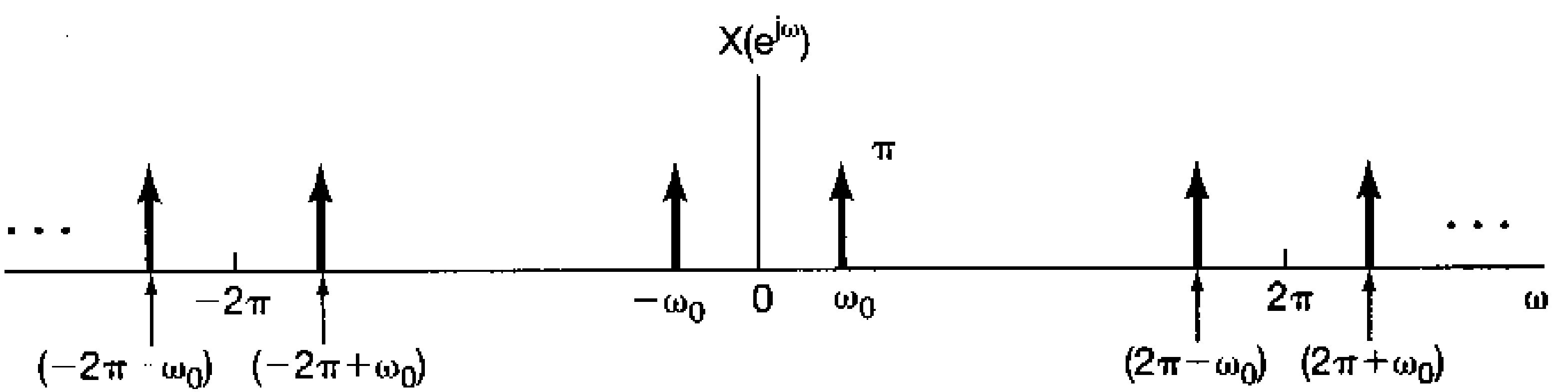
Refer to Fig. 5.9 at page 370

Ex. 5.5)

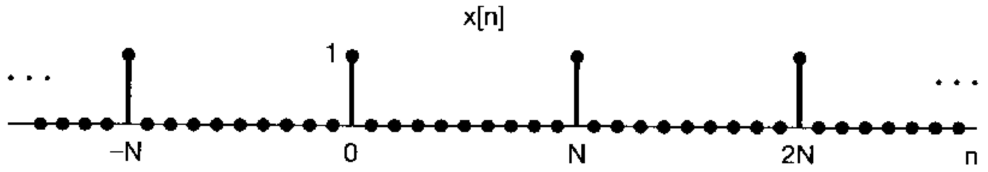
$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \quad \text{with} \quad \omega_0 = \frac{2\pi}{5}$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \qquad (-\pi \le \omega < \pi)$$



Ex. 5.6) $x[n]=\sum_{k=-\infty}^{+\infty}\delta[n-kN]$



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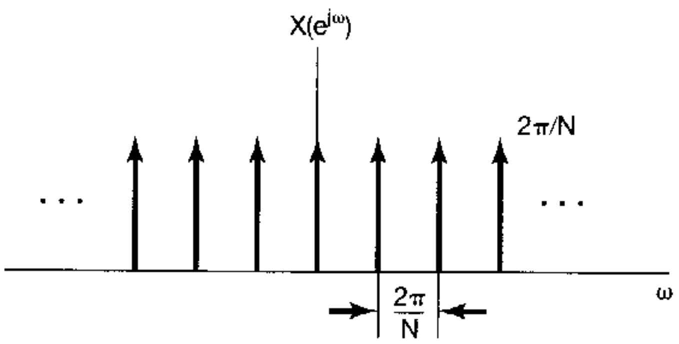


(b)

$$a_k=\frac{1}{N}\sum_{n=<N>}x[n]e^{-jk(2\pi/N)n}$$

$$=\frac{1}{N}\sum_{n=<N>}\sum_{l=-\infty}^{+\infty}\delta[n-lN]e^{-jk(2\pi/N)n}$$

$$=\frac{1}{N}\sum_{l=-\infty}^{+\infty}\sum_{n=<N>}\delta[n-lN]e^{-jk(2\pi/N)n}$$

$$=\frac{1}{N}\sum_{l=-\infty}^{+\infty}\sum_{n=0}^{N-1}\delta[n-lN]e^{-jk(2\pi/N)n}$$

$$=\frac{1}{N}$$

$$X(e^{j\omega})=\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$$

5.3 Properties of the discrete-time Fourier transform

5.3.1 Periodicity of the discrete-time Fourier transform

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

5.3.2 Linearity of the Fourier transform

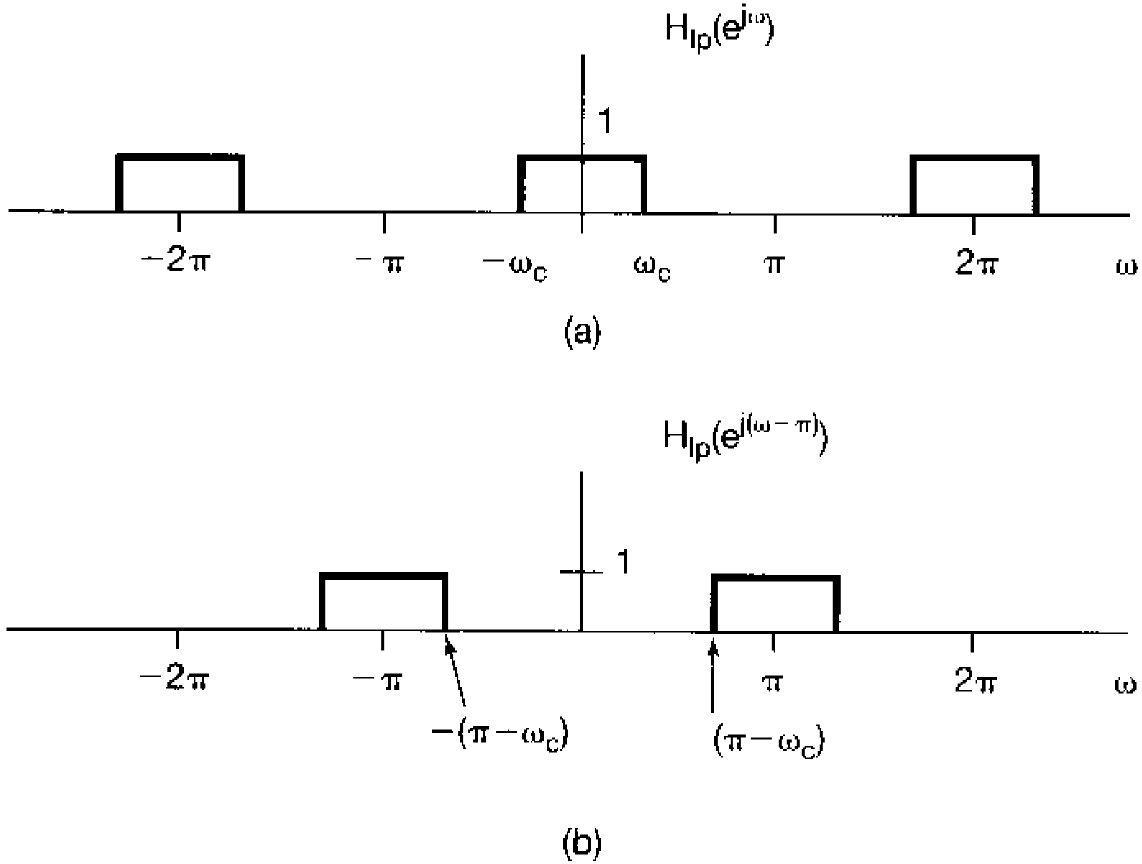
$$ax_1[n]+bx_2[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} aX_1(e^{j\omega})+bX_2(e^{j\omega})$$

5.3.3 Time shifting and frequency shifting

$$x[n-n_0] \stackrel{\mathfrak{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

Ex. 5.7)



$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

$$h_{hp}(n) = e^{j\pi n} h_{lp}[n]$$
$$= (-1)^n h_{lp}[n]$$

5.3.4 Conjugation and conjugate symmetry

$$x^*[n] \stackrel{\mathfrak{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

$$X(e^{j\omega}) = X^*(e^{-j\omega}), \quad [x[n]: \text{real}] \quad \text{conjugate symmetry}$$

$$\mathcal{E}v\{x[n]\} \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\}$$

$$\mathcal{O}d\{x[n]\} \stackrel{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m\{X(e^{j\omega})\}$$

5.3.5 Differencing and accumulation

$$x[n] - x[n-1] \stackrel{\mathfrak{F}}{\longleftrightarrow} (1-e^{-j\omega})X(e^{j\omega})$$

$$y[n] = \sum_{m=-\infty}^{n} x[m] \longrightarrow y[n] - y[n-1] = x[n]$$

$$\sum_{m=-\infty}^{n} x[m] \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$