Lecture Note for Solid Mechanics

- Analysis of Stress -

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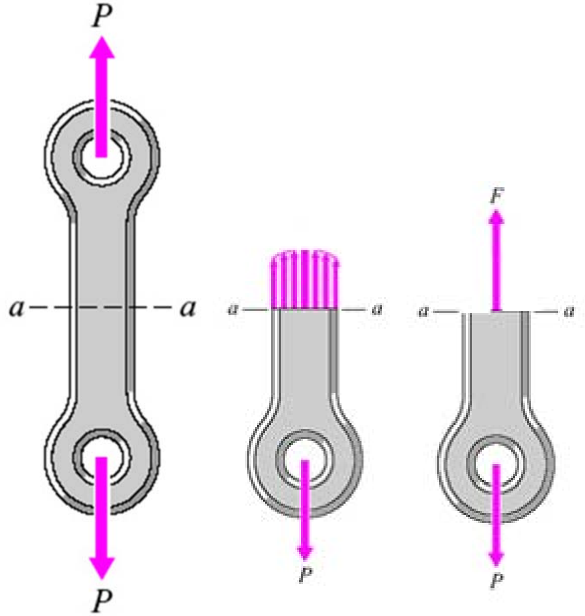
- **Text book : Mechanics of Materials, 6th ed.,**
  W.F. Riley, L.D. Sturges, and D.H. Morris, 2007.
- **Prerequisite : Knowledge of Statics, Basic Physics, Mathematics, etc.**

Analysis of stress

- Application of equilibrium equations
  - Determine the forces on a structure by its support and internal forces on the parts of the structure
- Objective of this chapter
  - Determine the internal effect of forces on the structure in order to understand the behavior of materials under the action of forces
- Design considerations : safety and economy
  - Calculate the internal force intensity and deformation on the parts
  - By knowing the properties of the material, engineers can establish the most effective size and shape of the parts

Analysis of stress

- Normal stress under axial loading
  - Axial force is defined as the force collinear with the centroidal axis of member
  - Stress (force intensity) = $\boxed{force/area}$
  - Normal stress : Normal force divided by the area over which the force is distributed



- Distribution of internal force
  - develops on exposed cross section
  - has a resultant F that is normal to the exposed cross section and is equal in magnitude of P
  - has a line of action collinear with the line of action of P

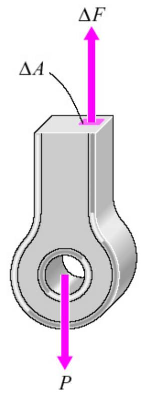
- Average intensity of internal force

$$\boxed{\sigma_{avg} = \frac{F}{A}}$$

+ : tensile
- : compressive

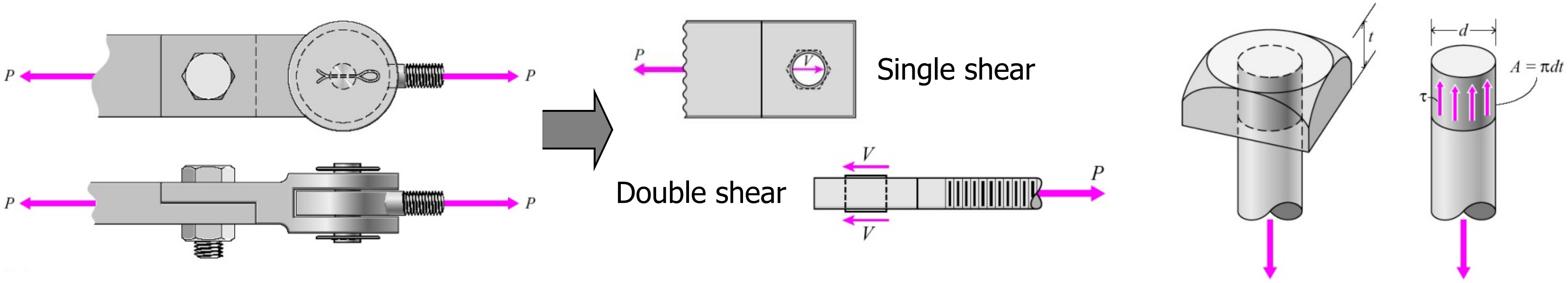
- Normal stress at a point on the cross section

$$\boxed{\sigma_{avg} = \frac{\Delta F}{\Delta A}} \Rightarrow \boxed{\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}}$$



Analysis of stress

- Shear stress in connections
  - The bolted and pinned connections are used to introduce the concept of shear stress
  - Shear stress : shear force divided by the area over which the force is distributed



- Average shear stress on the transverse cross section

$$\tau_{avg} = \frac{V}{A}$$

- Shear stress at a point on the transverse cross section

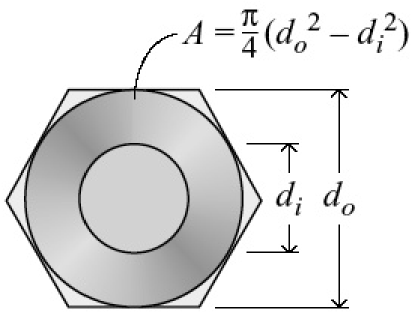
$$\tau_{avg} = \frac{\Delta V}{\Delta A} \Rightarrow \tau = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A}$$

- The shear stress can not be uniformly distributed over the area. Actual shear stress or maximum shear stress is different from the average stress.

Analysis of stress

- Bearing stress (compressive normal stress)
  - Bearing stress occurs on the contact surface between two interacting members and just names normal stress resulting from contact between two different bodies

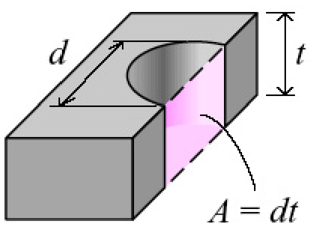
$$\boxed{\sigma_b = \frac{F}{A}}$$: Average bearing stress



(a)

- For the case of connection on the surfaces of contact between the head of the bolts and the top plate

$$\boxed{A = \frac{\pi}{4}(d_o^2 - d_i^2)}$$: annular area of the bolt or nut



(b)

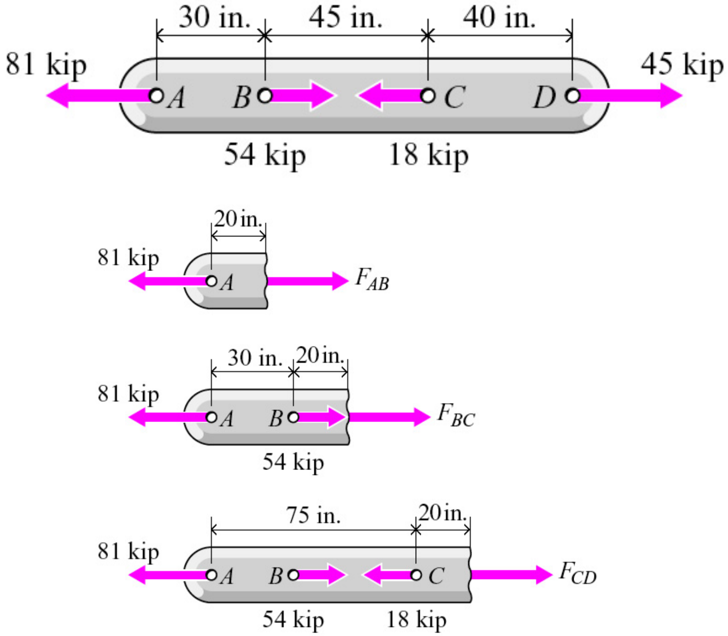
- For the case of connection where the shank of the bolt is pressed against the sides of the hole

$$\boxed{A = dt}$$: projected area (not actual contact area)

Analysis of stress

- Unit of stress
  - Force per unit area (FL^{-2})
  - psi or ksi (=1000psi) in English unit
  - Pa (N/m^2) or MPa (MN/m^2 or N/mm^2) in SI unit

(Example) Determine the normal stress in the bar (a) On a section 20 in. to the right of point A, (b) On a section 20 in. to the right section of point B, and (c) On a section 20 in. to the right of point C. Assume that the cross sectional area is 3 in^2.



(Sol)

$$\sum F = F_{AB} - 81 = 0 \qquad \Rightarrow F_{AB} = +81\ kip$$

$$\sum F = F_{BC} + 54 - 81 = 0 \qquad \Rightarrow F_{BC} = +27\ kip$$

$$\sum F = F_{CD} + 54 - 18 - 81 = 0 \qquad \Rightarrow F_{CD} = +45\ kip$$

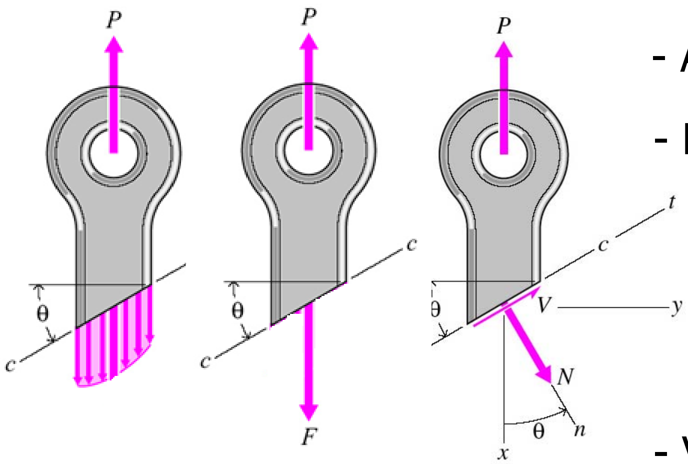
$$\sigma_{AB} = \frac{F_{AB}}{A} = \frac{+81}{3} = +27.0\ ksi$$

$$\sigma_{BC} = \frac{F_{BC}}{A} = \frac{+27}{3} = +9.0\ ksi$$

$$\sigma_{CD} = \frac{F_{CD}}{A} = \frac{+45}{3} = +15.0\ ksi$$

Analysis of stress

- Stresses on an inclined plane in an axially loaded member



- Resultant F of internal force is equal to the applied load P

- Average total stress $S_{avg} = \dfrac{F}{A}$

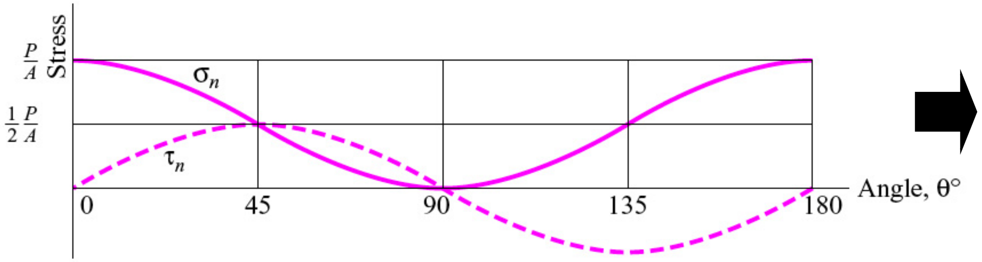
- Normal and shear stresses

$$N = P\cos\theta \qquad V = -P\sin\theta \qquad A_n = A/\cos\theta$$

$$\sigma_n = \frac{N}{A_n} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \frac{P}{2A}(1+\cos 2\theta)$$

$$\tau_n = \frac{V}{A_n} = \frac{-P\sin\theta}{A/\cos\theta} = \frac{-P}{A}\sin\theta\cos\theta = \frac{-P}{2A}\sin 2\theta$$

- Variations of normal and shear stresses



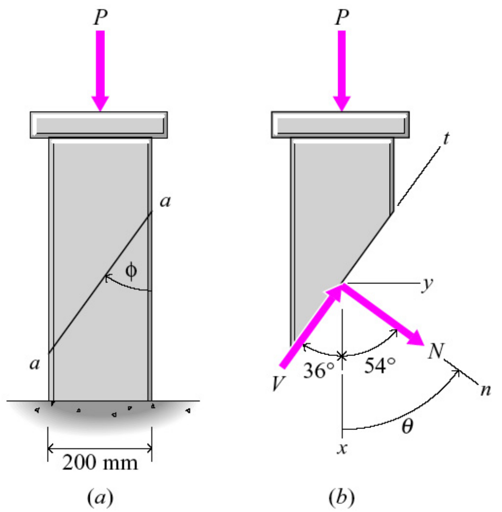
$$\sigma_{\max} = \frac{P}{A} \quad \text{at } 0^\circ \text{ and } 180^\circ$$

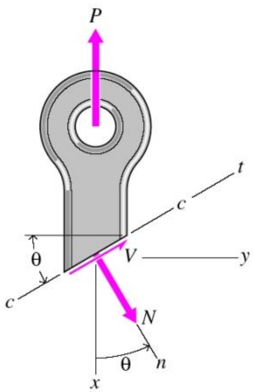
$$\tau_{\max} = \frac{P}{2A} \quad \text{at } 45^\circ \text{ and } 135^\circ$$

- Normal stress is either max. or min. on the planes for which shear stress is zero. Also, shear stress is zero on the planes of max. or min. normal stress.

Analysis of stress

(Example) The block has a 200x100-mm rectangular cross section. Normal stress on plane a-a is 12.0MPa (compression). If Φ is 36°, determine (a) load P (b) shear stress on plane a-a (c) maximum normal and shear stresses.





$$\sigma_n = \frac{P}{2A}(1+\cos 2\theta)$$
$$\tau_n = \frac{-P}{2A}\sin 2\theta$$

(Sol)

(a)
$$\sigma_n = \frac{N}{A_n} = \frac{P}{2A}\left(1+\cos 2\theta\right)$$
$$-12(10^6) = \frac{P}{2(200)(100)(10^{-6})}\left[1+\cos 2(+54)\right]$$
$$P = -694.7(10^3)N \cong 695\,kN(C)$$

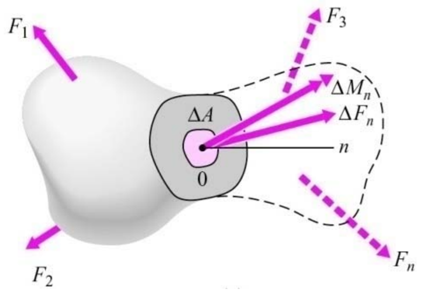
(b)
$$\tau_n = \frac{V}{A_n} = \frac{-P}{2A}\sin 2\theta$$
$$= \frac{-(-695)(10^3)}{2(200)(100)(10^{-6})}\sin 2(+54) = 16.52\,MPa$$

(c)
$$\sigma_{\max} = \frac{P}{A} = \frac{695(10^3)}{(200)(100)(10^{-6})} = 34.74(10^6)\frac{N}{m^2} = 34.7\,MPa$$
$$\tau_{\max} = \frac{P}{2A} = \frac{695\,(10^3)}{2(200)(100)(10^{-6})} = 17.37\,(10^6)\frac{N}{m^2} = 17.4MPa$$

Analysis of stress

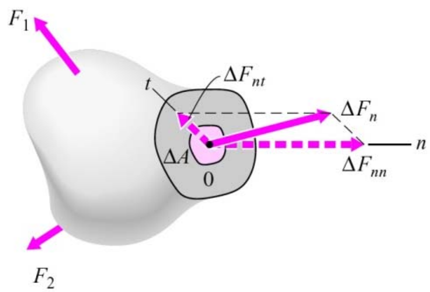
- Stress at a general point in an arbitrarily loaded member

- For any other plane through O the values of ΔF and ΔM could be different
- General concept of stress state at a point is needed



- Stress vector or resultant stress

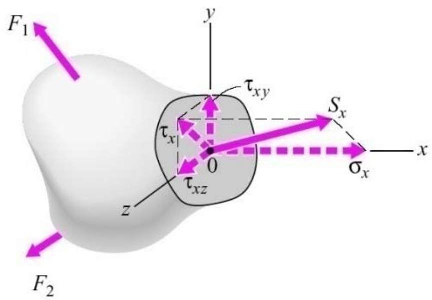
$$S_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$



- Normal stress and shear stress are defined as

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_{nn}}{\Delta A}$$

$$\tau_n = \lim_{\Delta A \to 0} \frac{\Delta F_{nt}}{\Delta A}$$

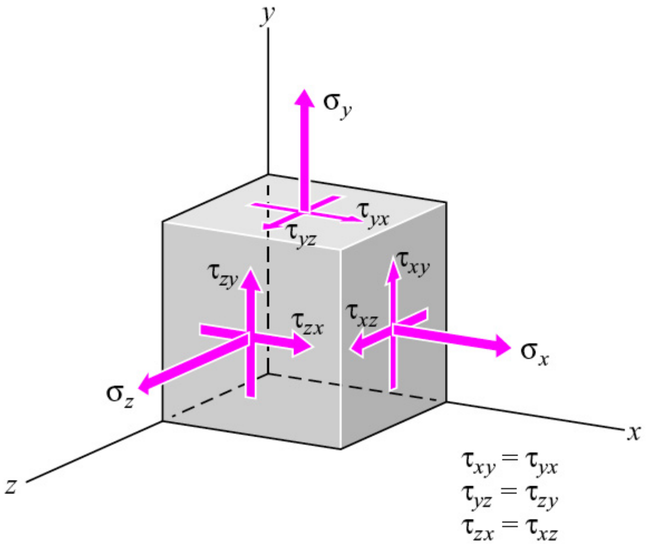


- Consider plane having an outward normal in the x-direction

$$\begin{bmatrix} \sigma_x \\ \tau_x \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_x \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix}$$

Analysis of stress

- Stress vector itself depends on the orientation of the plane with which it is associated
- Infinite number of planes can be passed through the point, resulting in an infinite number of stress vectors being associated with the point
- Specification of stresses on three mutually perpendicular planes is sufficient to completely describe the state of stress at the point



- Normal stress : single subscript indicates the plane on which the stress acts

σ_x, σ_y, σ_z → σ_{xx}, σ_{yy}, σ_{zz}

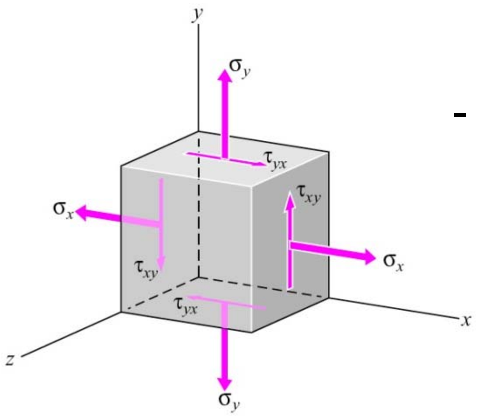
- Shear stress : first subscript indicates the plane on which the stress acts and second subscript indicates the coordinate axis to which the stress is parallel

τ_{xy}, τ_{yz}, τ_{zx}

Analysis of stress

- Two dimensional or plane stress

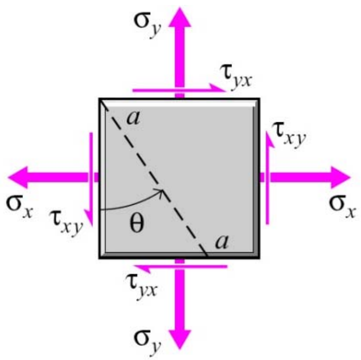
- Two parallel faces to be perpendicular to z-axis of the small element are assumed to be free of stress



- Two dimensional or plane stress problem

$$\sigma_z = \tau_{zx} = \tau_{zy} = \tau_{xz} = \tau_{yz} = \mathbf{0}$$

- Thin plates where z-dimension of the body is small and the z-components of force are zero



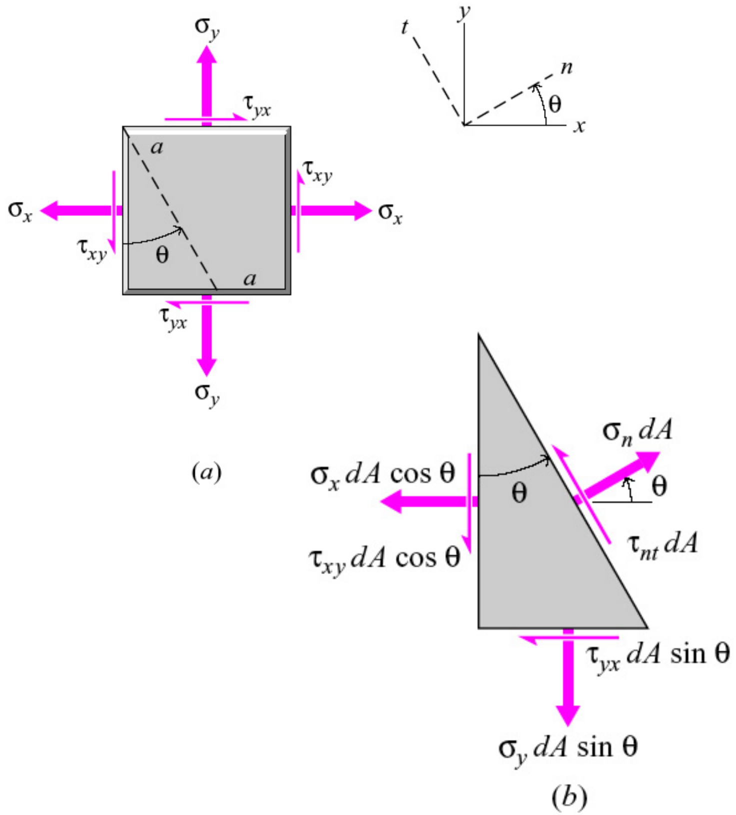
- The only components of stress in two dimensional problem

$$\sigma_x\ ,\ \ \sigma_y\ ,\ \ \tau_{xy} = \tau_{yx}$$

Analysis of stress

- Stress transformation equation for plane stress

- Equation relating the normal and shear stresses σ_n and τ_{nt} on an arbitrary plane (θ) and the known stresses σ_x, σ_y and τ_{xy} on the reference planes



$\sum F_n = 0$:

$$\sigma_n dA - \sigma_x (dA\cos\theta)\cos\theta - \sigma_y (dA\sin\theta)\sin\theta$$
$$-\tau_{yx}(dA\sin\theta)\cos\theta - \tau_{xy}(dA\cos\theta)\sin\theta = 0$$

$$\begin{aligned}\sigma_n &= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\ &= \sigma_x\frac{1+\cos 2\theta}{2} + \sigma_y\frac{1-\cos 2\theta}{2} + \tau_{xy}\sin 2\theta \\ &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\end{aligned}$$

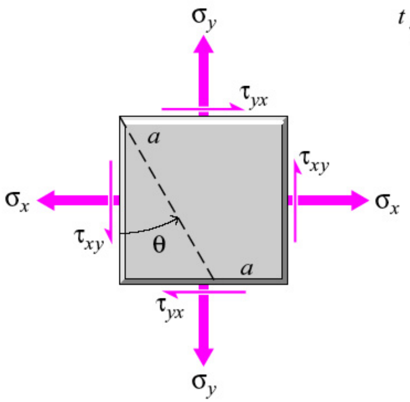
$\sum F_t = 0$:

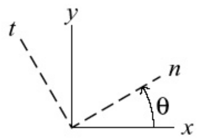
$$\tau_{nt} dA + \sigma_x (dA\cos\theta)\sin\theta - \sigma_y (dA\sin\theta)\cos\theta$$
$$-\tau_{xy}(dA\cos\theta)\cos\theta + \tau_{yx}(dA\sin\theta)\sin\theta = 0$$

$$\begin{aligned}\tau_{nt} &= -(\sigma_x-\sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \\ &= -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\end{aligned}$$

Analysis of stress

- sign convention for stress transformation equation

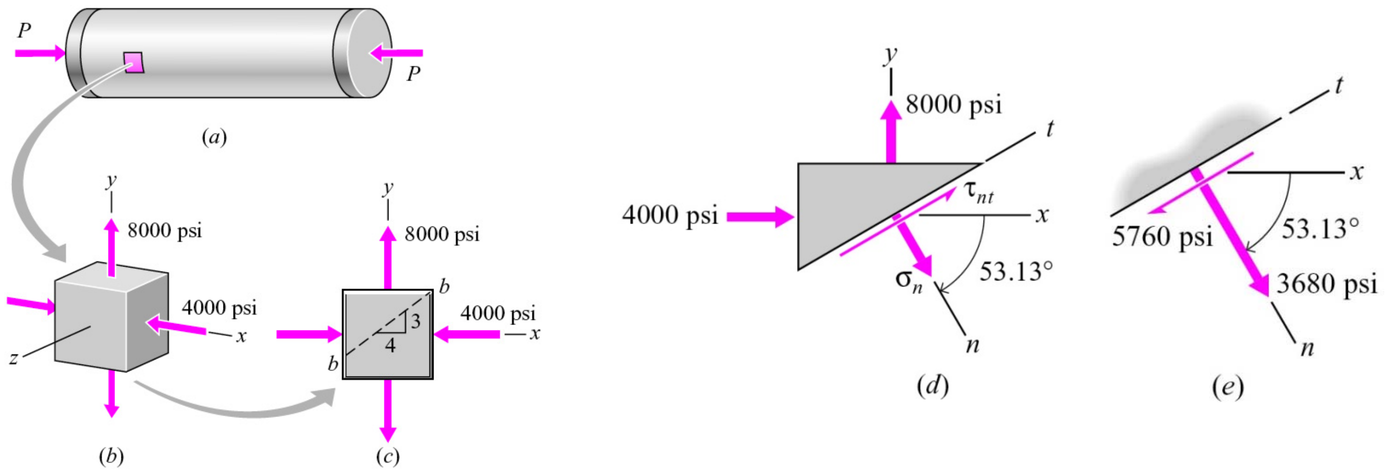




- Tensile normal stresses are positive; compressive normal stresses are negative. The sign of a normal stress is independent of the coordinate system.
- Shear stress is positive if it points in the positive direction of the coordinate axis of the second subscript when it is acting on a surface whose outward normal is in a positive direction of the coordinate axis of the first subscript. The sign of a shear stress depends on the coordinate system.
- Angle measured counterclockwise from the reference positive x-axis is positive.
- (n, t, z) axes have the same order and form a right-hand coordinate system as (x, y, z) axes.

Analysis of stress

(Example) At a point on the outside surface of a thin walled pressure vessel, stresses are 8000 psi(T) and zero shear on a horizontal and 4000 psi(C) and zero shear on a vertical plane. Determine the stresses at this point on plane b-b having a slope of 3 vertical to 4 horizontal.

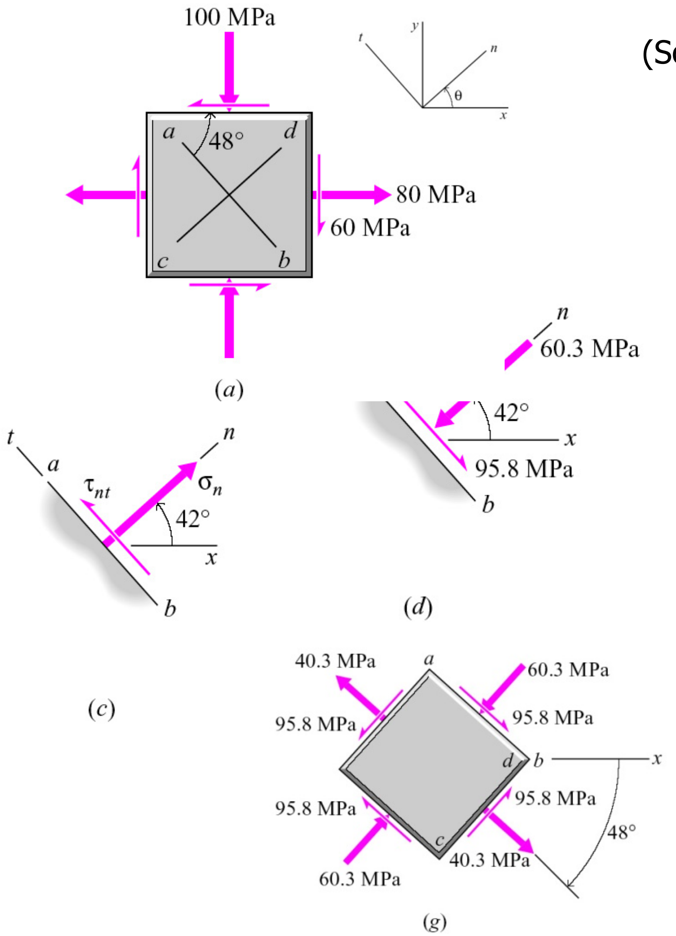


(Sol)

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$= (-4000)\cos^2(-53.13) + (+8000)\sin^2(-53.13) + 0$$
$$= +3680\ psi = 3680\ psi\,(T)$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -[(-4000) - (+8000)]\sin(-53.13)\cos(-53.13) + 0$$
$$= -5760\ psi = 5760\ psi\,(C)$$

(Example) Determine (a) normal and shear stresses on plane a-b, (b) normal and shear stresses on plane c-d, which is perpendicular to plane a-b, (c) show the stresses on planes a-b and c-d using a small element.



(Sol) $\boxed{\sigma_x = 80\,MPa,\ \sigma_y = -100\,MPa,\ \tau_{xy} = \tau_{yx} = -60\,MPa}$

(a) $\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$

$$= 80\cos^2(+42) + (-100)\sin^2(+42) + 2(-60)\sin(+42)\cos(+42)$$

$$= -60.26\,MPa \cong 60.3\,MPa\ (C)$$

$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$

$$= -[80-(-100)]\sin(+42)\cos(+42) + (-60)[\cos^2(+42) - \sin^2(+42)]$$

$$= -95.78\,MPa \cong -95.8\,MPa$$

(b) $\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$

$$= 80\cos^2(-48) + (-100)\sin^2(-48) + 2(-60)\sin(-48)\cos(-48)$$

$$= +40.26\,MPa \cong 40.3\,MPa\ (T)$$

$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$

$$= -[80-(-100)]\sin(-48)\cos(-48) + (-60)[\cos^2(-48) - \sin^2(-48)]$$

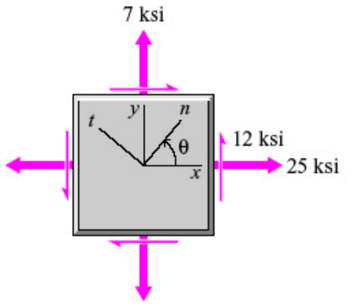
$$= +95.78\,MPa \cong +95.8\,MPa$$

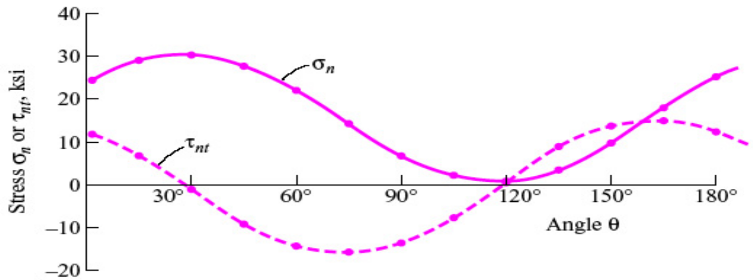
(c) Since planes a-b and c-d through the point are perpendicular, the stresses on these planes may be drawn on an element as figure (g).

Analysis of stress

- Principal stresses and maximum shear stress - plane stress

  - Transformation equations for plane stress provide a means for determining the normal stress and the shear stress on different planes through a point.
  - For design purposes, critical stresses at the point are the maximum normal and shear stresses





$$\text{at } \theta = 0^o \qquad \sigma_n = \sigma_x, \quad \tau_{nt} = \tau_{xy}$$

$$\text{at } \theta = 90^o \qquad \sigma_n = \sigma_y, \quad \tau_{nt} = -\tau_{yx} = -\tau_{xy}$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

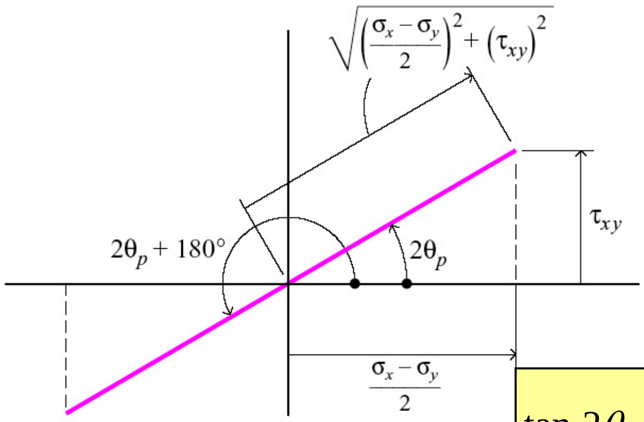
$$\frac{d\sigma_n}{d\theta} = 0 = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta$$

$$\tan 2\theta_p = \frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Shear stress is zero on planes experiencing maximum and minimum values of normal stress.
- Planes free of shear stress are the principal planes
- Normal stresses on principal planes are principal stresses

- The above relation gives the orientations of two principal planes.
- Principal planes are normal to each other. (Two values of $2\theta_p$ differ by 180^o)
- A third principal plane for the plane stress state has outward normal in z-direction.

Analysis of stress



$$\tan 2\theta_p = \frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- For a given set of stress values, there are two values of $2\theta_p$ differing by 180°
- The above relation gives the orientations of two principal planes which are normal to each other. (Two values of θ_p differ by 90°)
- A third principal plane for the plane stress state has outward normal in z-direction.

If τ_{xy} and (σ_x - σ_y) have the same sign ($\tan 2\theta_p$ is positive),

⇨ $2\theta_p$ is between 0° and 90° with the other value 180° greater

θ_p is between 0° and 45° with the other value 90° greater (θ_p and θ_p+90°)

- Both $\sin 2\theta_p$ and $\cos 2\theta_p$ are positive

$$\cos 2\theta_p = \frac{\frac{\sigma_x-\sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}} \qquad \sin 2\theta_p = \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

- Both $\sin 2\theta_p$ and $\cos 2\theta_p$ are negative

$$\cos 2\theta_p = \frac{-\frac{\sigma_x-\sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}} \qquad \sin 2\theta_p = \frac{-\tau_{xy}}{\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}}$$

⇨

$$\sigma_{p1,\,p2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$\sigma_{p3} = \sigma_z = 0$$

Analysis of stress

Maximum in-plane shear stress occurs on planes by the specified values of θ

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + 2\tau_{xy}\cos 2\theta$$

$$\frac{d\tau_{nt}}{d\theta} = 0 = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta$$

$$\tan 2\theta_\tau = \frac{\sin 2\theta_\tau}{\cos 2\theta_\tau} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\tan 2\theta_p = \frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

(a) Two tangents are negative reciprocals

(b) Two angles $2\theta_p$ and $2\theta_\tau$ differ 90°, and θ_p and θ_τ differ 45° apart

(c) Planes on which maximum in-plane shear stress occur are 45° from the principal planes

$$\tau_p = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{\sigma_{p1} - \sigma_{p2}}{2}$$

- There are two perpendicular planes of maximum in-plane shear stress, having the same magnitude but opposite signs
- Maximum shear stress is equal in magnitude to one-half the difference between the two principal stresses

Analysis of stress

Relation between the principal stresses and the maximum in-plane shear stress

$$\sigma_{p1} - \sigma_{p2} = \left(\frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \right) - \left(\frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \right)$$

$$= 2\sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 2\tau_p$$

$$\Rightarrow \quad \tau_p = \frac{\sigma_{p1} - \sigma_{p2}}{2}$$

- There are two perpendicular planes of maximum in-plane shear stress, having the same magnitude but opposite signs
- Maximum shear stress is equal in magnitude to one-half the difference between the two principal stresses

Relation between the principal stresses and the normal stresses on the orthogonal planes

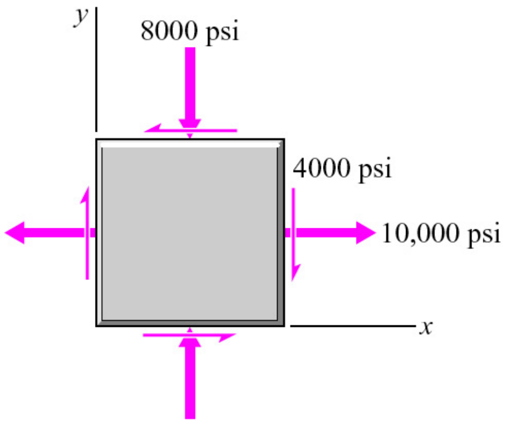
$$\sigma_{p1} + \sigma_{p2} = \left(\frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \right) + \left(\frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \right)$$

$$= 2\left(\frac{\sigma_x + \sigma_y}{2} \right) = \sigma_x + \sigma_y$$

Analysis of stress

(Example) (a) Determine principal stresses and maximum shearing stress (b) locate planes on which these stresses act and show the stresses on a complete sketch.

(Sol) (a) $\boxed{\sigma_x = 10000\,psi\,,\ \ \sigma_y = -8000\,psi,\ \ \tau_{xy} = -4000\,psi}$



$$\sigma_{p1,\,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{10000 + (-8000)}{2} \pm \sqrt{\left(\frac{10000 - (-8000)}{2}\right)^2 + (-4000)^2}$$

$$= 1000 \pm 9849$$

$$\sigma_{p1} = +10850\,psi = 10850\,psi\ (T)$$

$$\sigma_{p2} = -8850\,psi = 8850\,psi\ (C)$$

$$\sigma_{p3} = \sigma_z = 0\,psi$$

Maximum in-plane shear stress

$$\tau_p = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \pm\sqrt{\left(\frac{(10000) - (-8000)}{2}\right)^2 + (-4000)^2}$$

$$= \pm 9850\,psi$$

Maximum shear stress

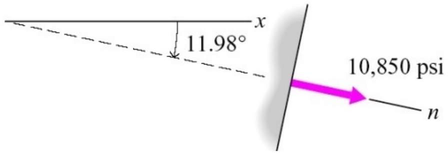
$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2}$$

$$= \frac{10850 - (-8850)}{2} = 9850\,psi$$

(b) $\tan 2\theta_p = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y} = \dfrac{2(-4000)}{10000 - (-8000)} = -0.444$

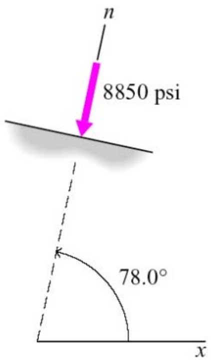
$2\theta_p = -23.96^\circ, +156.04^\circ, \cdots$

$\theta_p = -11.98^\circ, +78.02^\circ, \cdots$



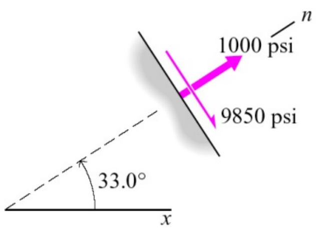
when $\theta_p = -11.98^\circ$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$= 10000\cos^2(-11.98) + (-8000)\sin^2(-11.98) + 2(-4000)\sin(-11.98)\cos(-11.98)$$
$$= \sigma_{p1} = +10850\ psi = 10850\ psi\ (T)$$



when $\theta_p = +78.02^\circ$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$= 10000\cos^2(+78.02) + (-8000)\sin^2(+78.02) + 2(-4000)\sin(+78.02)\cos(+78.02)$$
$$= \sigma_{p2} = -8850\ psi = 8850\ psi\ (C)$$



when $\theta_\tau = -11.98^\circ + 45^\circ = +33.02^\circ$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$= 10000\cos^2(+33.02) + (-8000)\sin^2(+33.02) + 2(-4000)\sin(+33.02)\cos(+33.02)$$
$$= +1000\ psi = 1000\ psi\ (T)$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -[10000 - (-8000)]\sin(+33.02)\cos(+33.02)$$
$$+ (-4000)[\cos^2(+33.02) - \sin^2(+33.02)]$$
$$= -9850\ psi$$

Analysis of stress

$$when \quad \theta_\tau = +33.02^\circ + 90^\circ = 123.02^\circ$$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= 10000\cos^2(+123.02^\circ) + (-8000)\sin^2(+123.02^\circ) + 2(-4000)\sin(+123.02^\circ)\cos(+123.02^\circ)$$

$$= +1000\ psi = 1000\ psi\ (T)$$

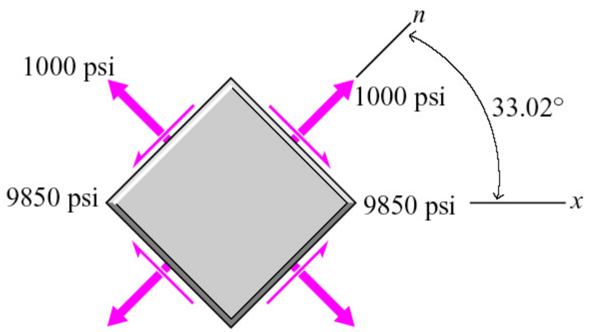
$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -[10000 - (-8000)]\sin(+123.02^\circ)\cos(+123.02^\circ)$$

$$+ (-4000)[\cos^2(+123.02^\circ) - \sin^2(+123.02^\circ)]$$

$$= +9850\ psi$$

- Normal stresses on the perpendicular planes of maximum shear stresses are equal both in magnitude and sign
- Shear stress have equal magnitude but opposite sign



Analysis of stress

- Mohr's circle for plane stress

Graphic interpretation of transformation equations for plane stress

$$\sigma_n - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

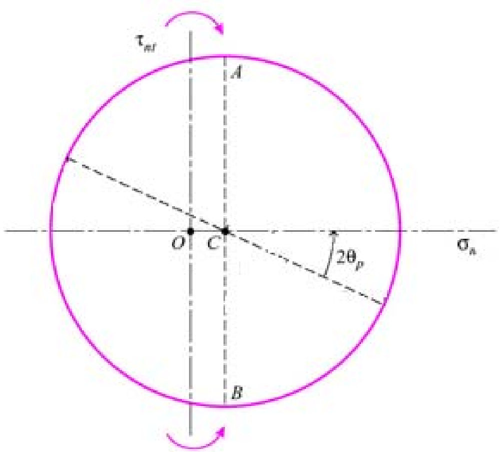
$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\left(\sigma_n - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \left(\tau_{nt}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2$$

$$Center \;\; is \;\; \left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$$

$$Radius \;\; is \;\; R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

- Coordinates of each point on the circle represent the normal and shear stresses on one plane through the stresses point
- Angular position of the radius to the point gives the orientation of the plane



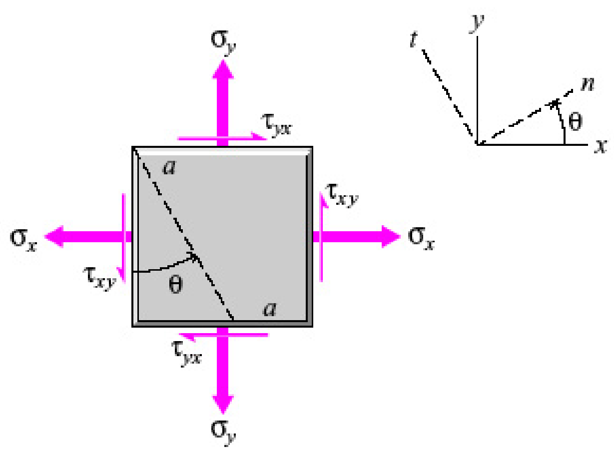
How to draw a Mohr's circle

- Normal stresses are plotted as horizontal coordinates (tensile stresses plotted to the right of the origin)
- Shear stresses are plotted as vertical coordinates (clockwise rotation of stress element plotted above σ-axis)

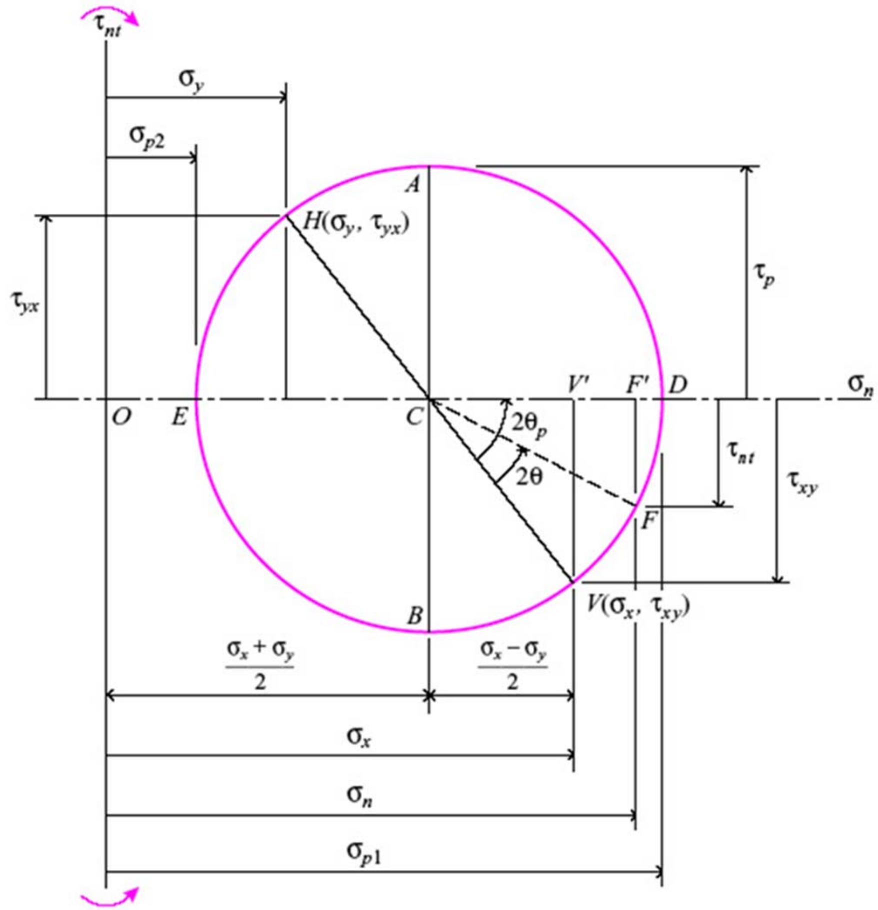
Analysis of stress

Procedures to draw a Mohr's circle

1. Choose a set of x-y reference axes
2. Identify stresses and list them with proper sign
3. Draw a set of coordinate axes
4. Plot $(\sigma_x, -\tau_{xy})$ and label it point V (vertical plane)
5. Plot (σ_x, τ_{yx}) and label it point H (horizontal plane)
6. Draw line between V and H
7. Establish center C and radius R
8. Draw a circle

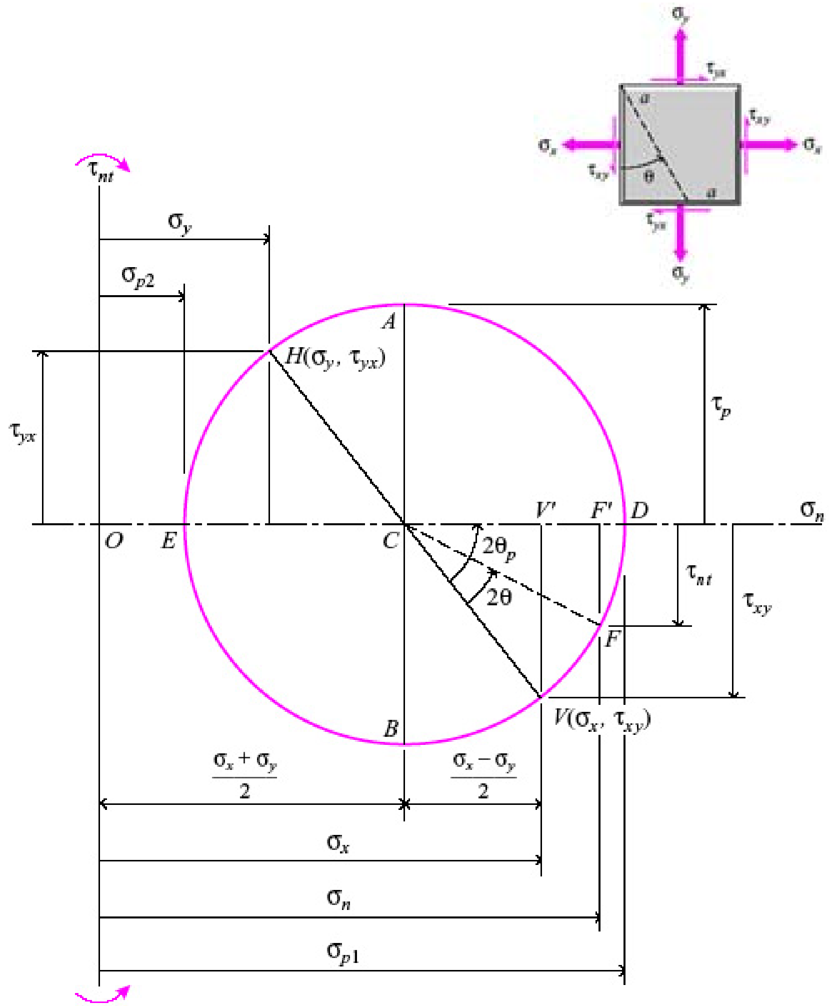


In the case of the σ_x greater than σ_y



Analysis of stress

- Mohr's Circle for plane stress



V : stresses on the vertical plane
H : stresses on the horizontal plane
C : center of circle
Line CV : vertical plane from which angle is measured

$$OF' = OC + CF\cos(2\theta_p - 2\theta)$$
$$= OC + CV\cos 2\theta_p \cos 2\theta + CV\sin 2\theta_p \sin 2\theta$$

Using $CV\cos 2\theta_p = CV' = (\sigma_x - \sigma_y)/2$

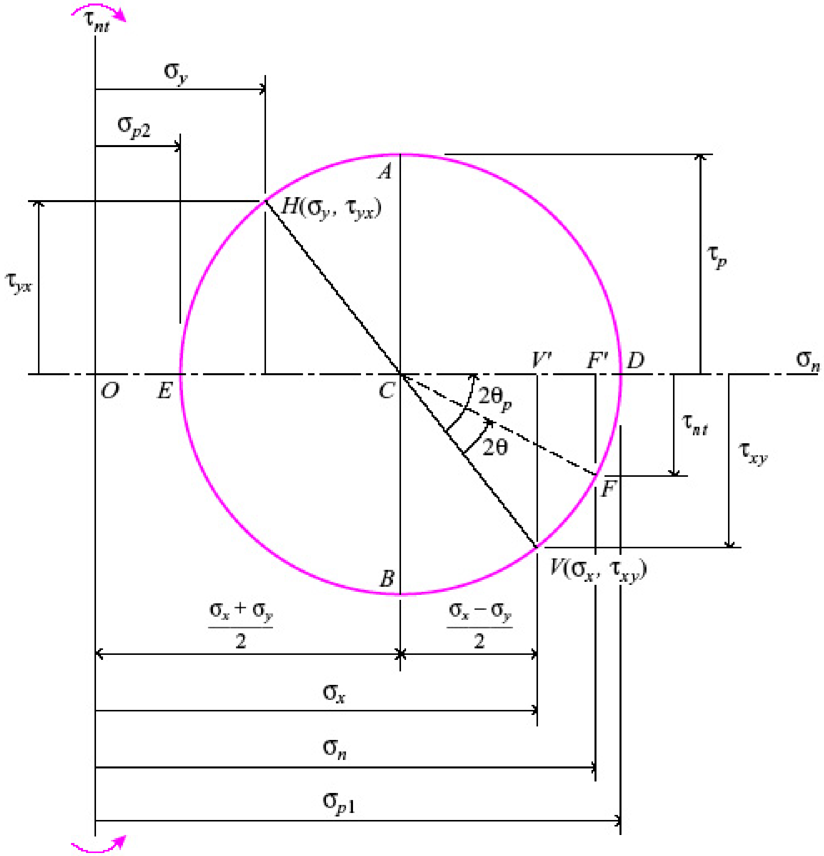
$CV\sin 2\theta_p = VV' = \tau_{xy}$

$OC = (\sigma_x + \sigma_y)/2 = \sigma_{avg}$

$$OF' = OC + CV'\cos 2\theta + VV'\sin 2\theta$$
$$= \frac{\sigma_x + \sigma_x}{2} + \frac{\sigma_x - \sigma_x}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$FF' = CF\sin(2\theta_p - 2\theta)$$
$$= CV\sin 2\theta_p \cos 2\theta - CV\cos 2\theta_p \sin 2\theta$$
$$= VV'\cos 2\theta - CV'\sin 2\theta$$
$$= \tau_{xy}\cos 2\theta - \frac{\sigma_x - \sigma_x}{2}\sin 2\theta$$

Analysis of stress



$$\sigma_{p1} = OD = OC + CD = OC + CV$$

$$= \frac{\sigma_x + \sigma_x}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_x}{2}\right)^2 + (\tau_{xy})^2}$$

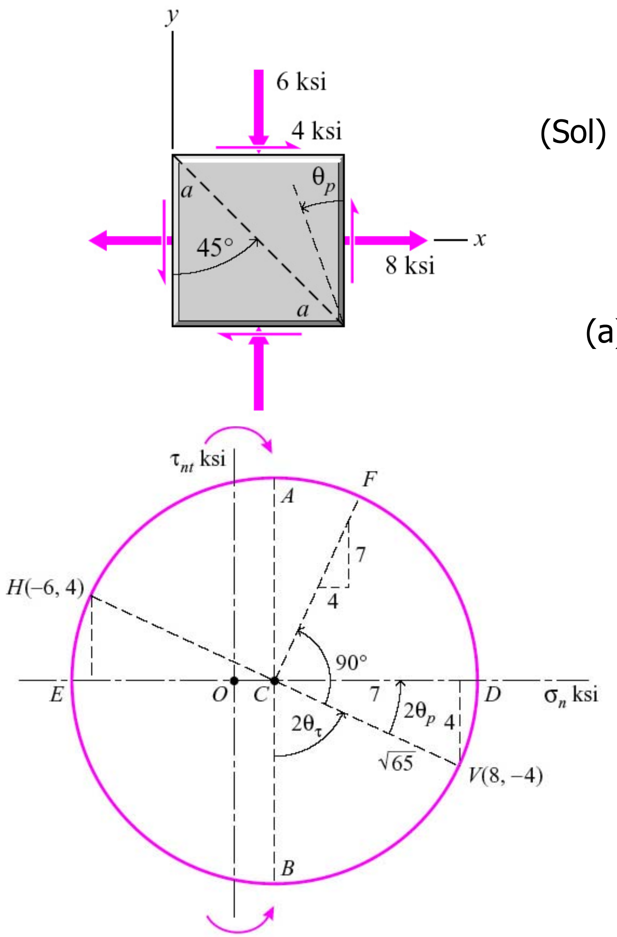
$$\tau_p = CA = CB = \sqrt{\left(\frac{\sigma_x - \sigma_x}{2}\right)^2 + (\tau_{xy})^2}$$

- If the two nonzero principal stresses have the same sign, the maximum shear stress at the point will not be in the plane of the applied stresses

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

- The angle between the vertical plane and one of the principal planes is θ_p
- All angle on Mohr's circle are twice the corresponding angles for the actual stressed body

(Example 2-11) Determine (a) principal stresses and maximum shearing stress (b) normal and shearing stresses on a-a plane.



(Sol)

$$V:(8,-4) \quad H:(-6,4) \quad C:(1,0)$$
$$CV = \sqrt{7^2+4^2} = 8.06\,ksi$$

(a) $\sigma_{p1} = OD = 1 + 8.06 = +9.06\,ksi = 9.06\,ksi\ (T)$

$\sigma_{p2} = OE = 1 - 8.06 = -7.06\,ksi = 7.06\,ksi\ (C)$

$\sigma_{p3} = \sigma_z = 0$

Since two principal stresses have opposite signs

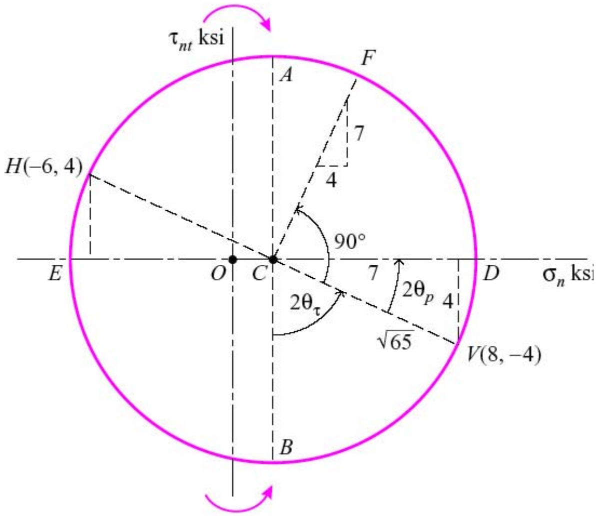
$\tau_p = \tau_{max} = CA = CB = 8.06\,ksi$

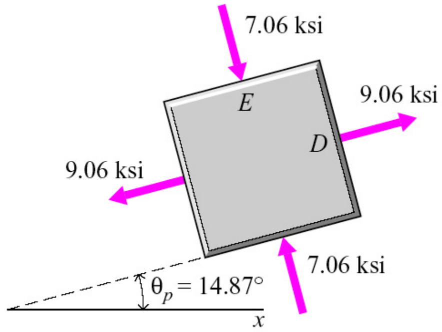
The principal planes are represented by lines CD and CE

$$\tan 2\theta_p = \frac{4}{7} = 0.5714$$

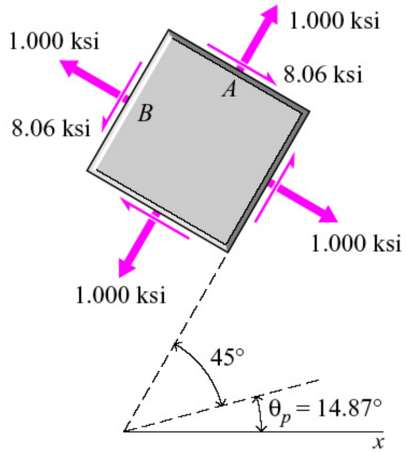
$$2\theta_p = +29.74^\circ \quad or \quad \theta_p = +14.87^\circ$$

Analysis of stress



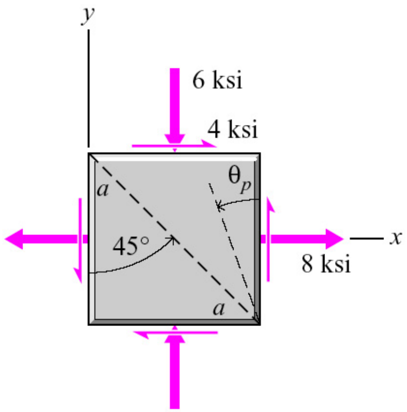


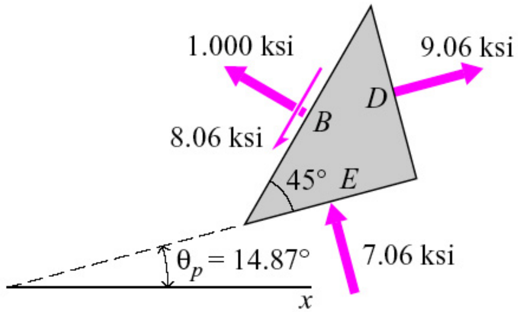
(c)



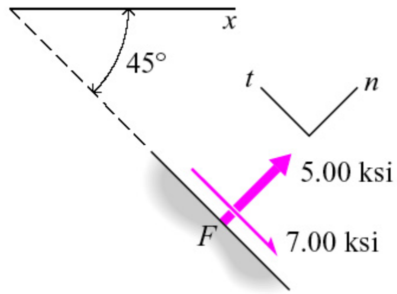
(d)

- Two orthogonal surfaces are sufficient to completely specify the principal stresses
- One of the surfaces is required to completely specify the maximum shear stress





(e)



(f)

(Homework)

(2-8), (2-18), (2-37), (2-58), (2-74), (2-93)