Lecture Note for Solid Mechanics

- Stress-Strain Relationships and Material Properties -

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- **Text book : Mechanics of Materials, 6th ed.,**

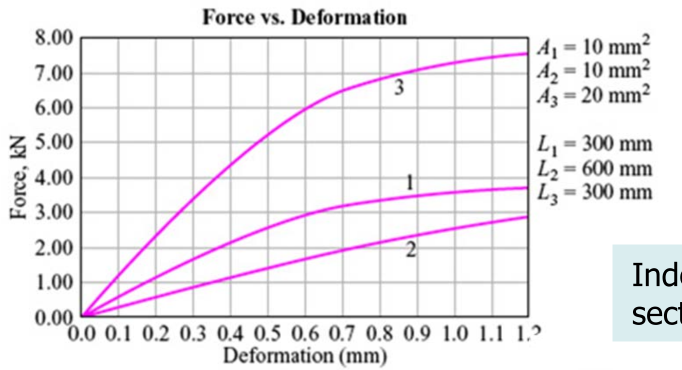
  W.F. Riley, L.D. Sturges, and D.H. Morris, 2007.

- **Prerequisite : Knowledge of Statics, Basic Physics, Mathematics, etc.**

Stress-Strain Relations and Material Properties

- Stress-strain diagrams
  - It is necessary to relate loads and temperature changes on the structure to the deformations produced by the loads and temperature changes
  - Relationship between load and deformation depends on dimension of member and type of material

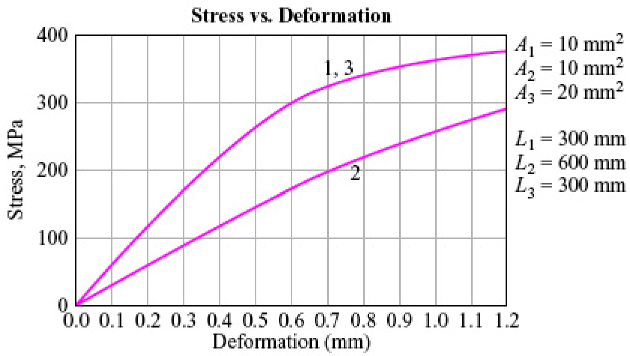
Dependent of cross sectional area & length



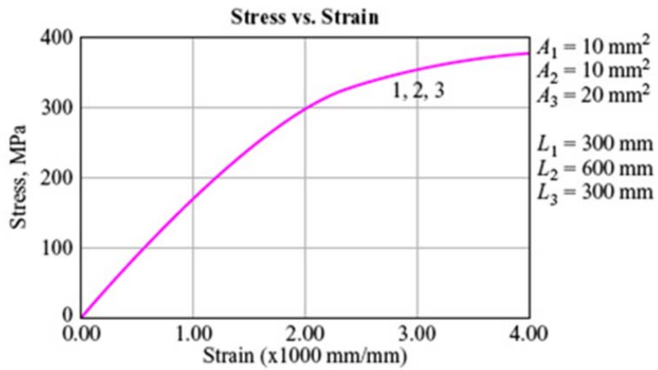
(*a*)

Independent of cross sectional area

Independent of length cross sectional area and length



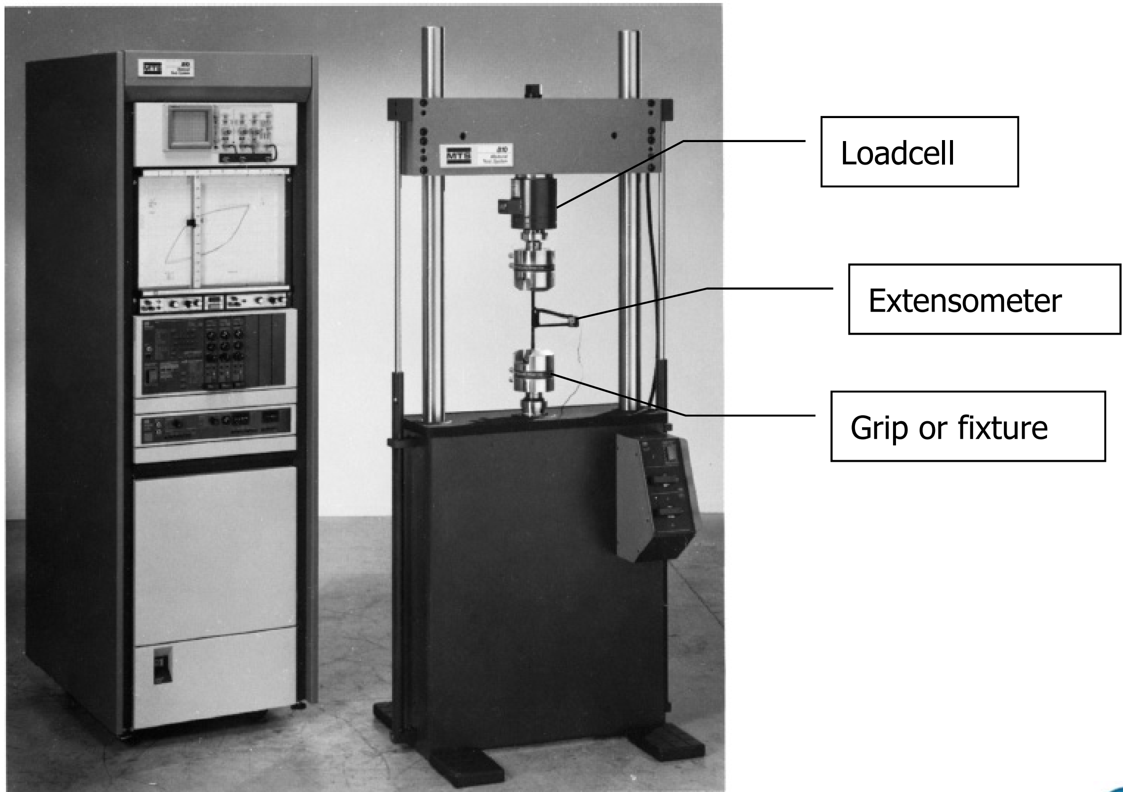
(*b*)



(*c*)

Stress-Strain Relations and Material Properties

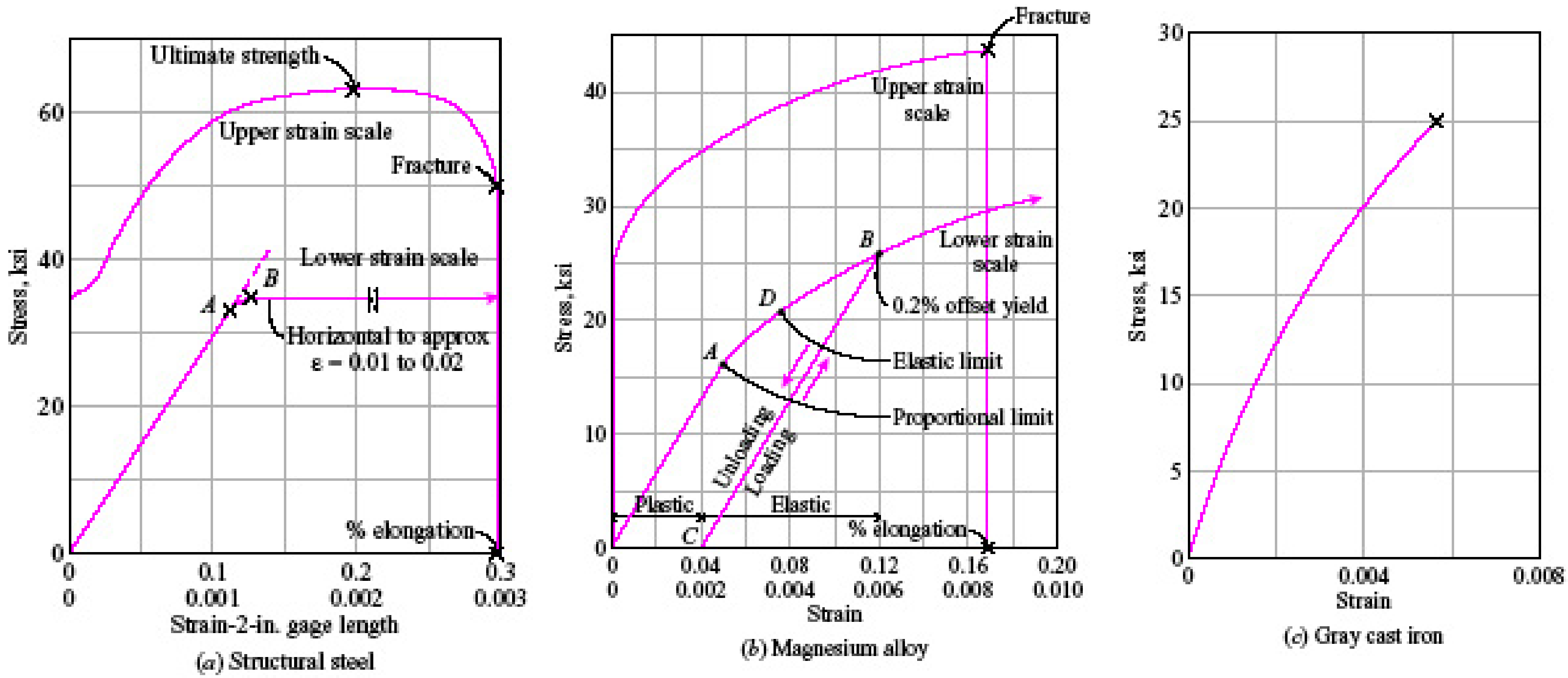
- The tensile test
  - Data for stress-strain curves are obtained by applying an axial load to a specimen and measuring the load and deformation simultaneously
  - Stress is obtained by dividing the load by the initial area (nominal stress) or the actual area (true stress)



Stress-Strain Relations and Material Properties

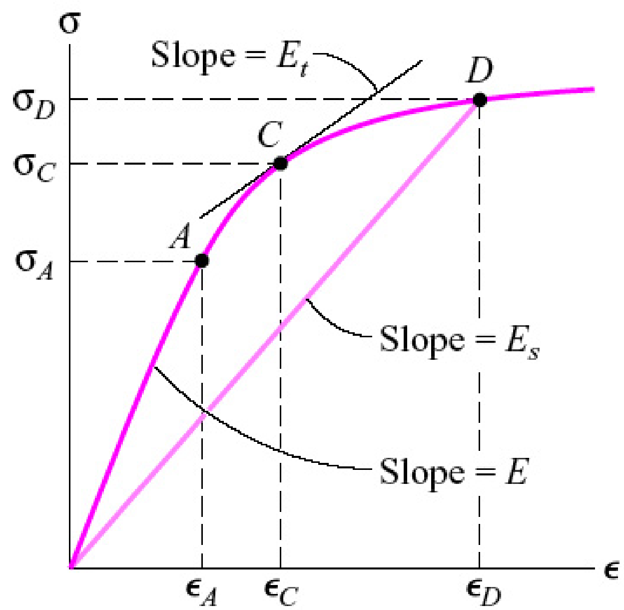
- Strain measurement
  - Instruments for measuring strain : strain gages or extensometers
  - Nominal strain is computed on the initial length of the specimen
  - True strain is computed on the actual length of the specimen

- Example of stress-strain curves



Stress-Strain Relations and Material Properties

- Modulus of elasticity (Young's modulus)
  - The slope of straight line portion of stress-strain curve
  - is used to measure the stiffness of material
  - Young's modulus (Modulus of elasticity) : $\sigma = E\varepsilon$
  - Shear modulus (Modulus of rigidity) : $\tau = G\gamma$



- Proportional limit (σ_A)

  The maximum stress for which stress and strain are proportional

- Young's modulus (E)

  The slope of straight line portion of stress-strain curve

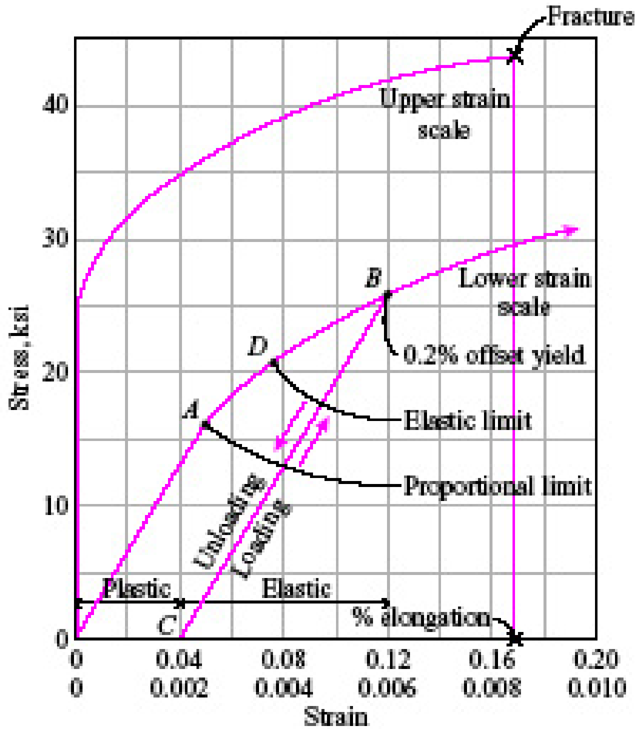
- Tangent modulus (E_t)

  The slope of stress-strain curve at a particular point beyond the proportional limit

- Secant modulus (E_s)

  The slope of stress-strain curve at any point

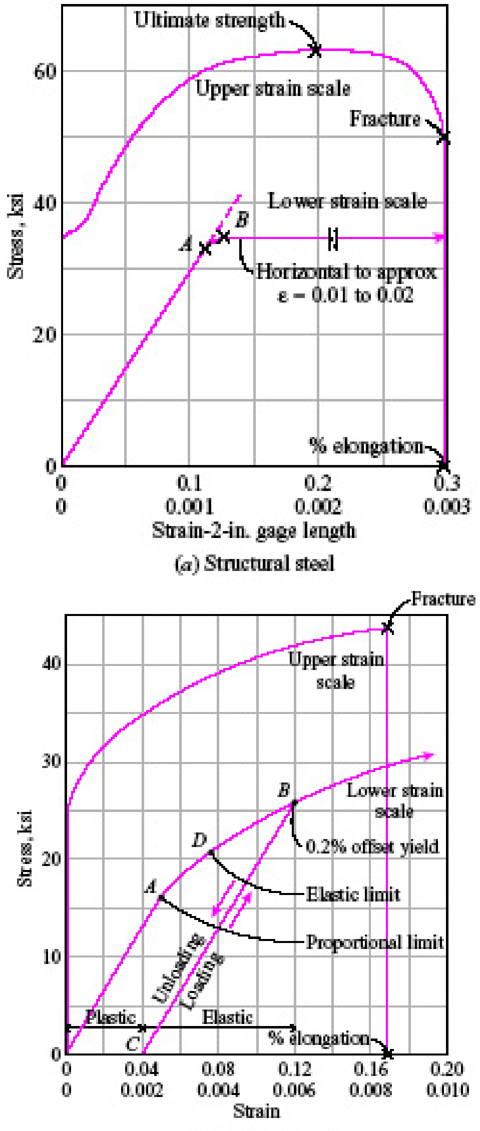
- Elastic limit
  - Elastic if strain resulting from loading disappears when the load is removed
  - Elastic limit (point D) is the maximum stress for which material acts elastically
  - Line BC is parallel to line OA
  - Precise value is difficult to obtain



(b) Magnesium alloy

- Work hardening (or strain hardening)
  - If loading again, stress-strain diagram follows the unloading curve until it reaches a stress a little less than the maximum stress during initial loading
  - The proportional limit (point B) is greater than that for the initial loading

Stress-Strain Relations and Material Properties

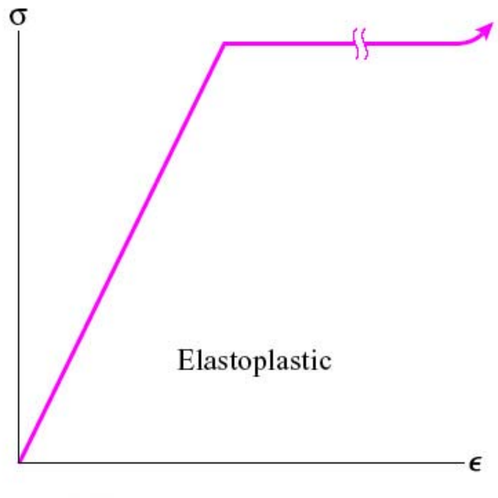


(a) Structural steel

(b) Magnesium alloy

- Yield point
  - The stress at which there is an appreciable increase in strain with no increase in stress
  - Low carbon steels
- Yield strength
  - The stress that induce a specified permanent set with 0.2 percent (0.002)
  - Stress indicated by intersection of CB and stress-strain diagram
- Ultimate strength
  - The maximum stress based on the original area developed in a material before rupture
  - The nominal stress decreases beyond the ultimate strength
  - The true stress continues to increase until rupture

Stress-Strain Relations and Material Properties



- Elastoplastic Materials
  - When the stress exceeds the proportional limit, empirical equations have been proposed relating between stress and strain
  - Plastic strain is 16-20 times the elastic strain at the proportional limit
- Ductility
  - Capacity for plastic deformation
  - Controls the amount of cold forming to which a material may be subjected
- Poisson's ratio
  - Ratio of the lateral strain to longitudinal strain

$$\nu = -\frac{\varepsilon_{lat}}{\varepsilon_{long}} = -\frac{\varepsilon_{l}}{\varepsilon_{a}} = -\frac{\varepsilon_{\perp}}{\varepsilon_{//}}$$

$$E = 2(1+\nu)G$$

(Example 4-1) A 100-kip axial load is applied to a 1x4x90-in rectangular bar. When loaded, the 4-in side measures 3.9986-in and the length has increased 0.09-in. determine Poisson's ratio, the modulus of elasticity, and the modulus of rigidity of the material.

(Sol)

$$\delta_{lat} = 3.9986 - 4 = -0.0014 \ in$$

$$\varepsilon_{lat} = \frac{\delta_{lat}}{L} = \frac{-0.0014}{4} = -0.00035$$

$$\varepsilon_{long} = \frac{\delta_{long}}{L} = \frac{0.09}{90} = 0.00100$$

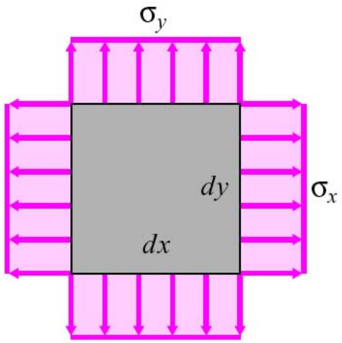
$$\sigma = \frac{P}{A} = \frac{100}{4 \times 1} = 25 \ ksi$$

$$Poisson's \ ratio : \quad \nu = -\frac{\varepsilon_{lat}}{\varepsilon_{long}} = -\frac{-0.00035}{0.00100} = 0.35$$

$$Modulus \ of \ elasticity : \quad E = \frac{\sigma}{\varepsilon} = \frac{25}{0.00100} = 25{,}000 \ ksi$$

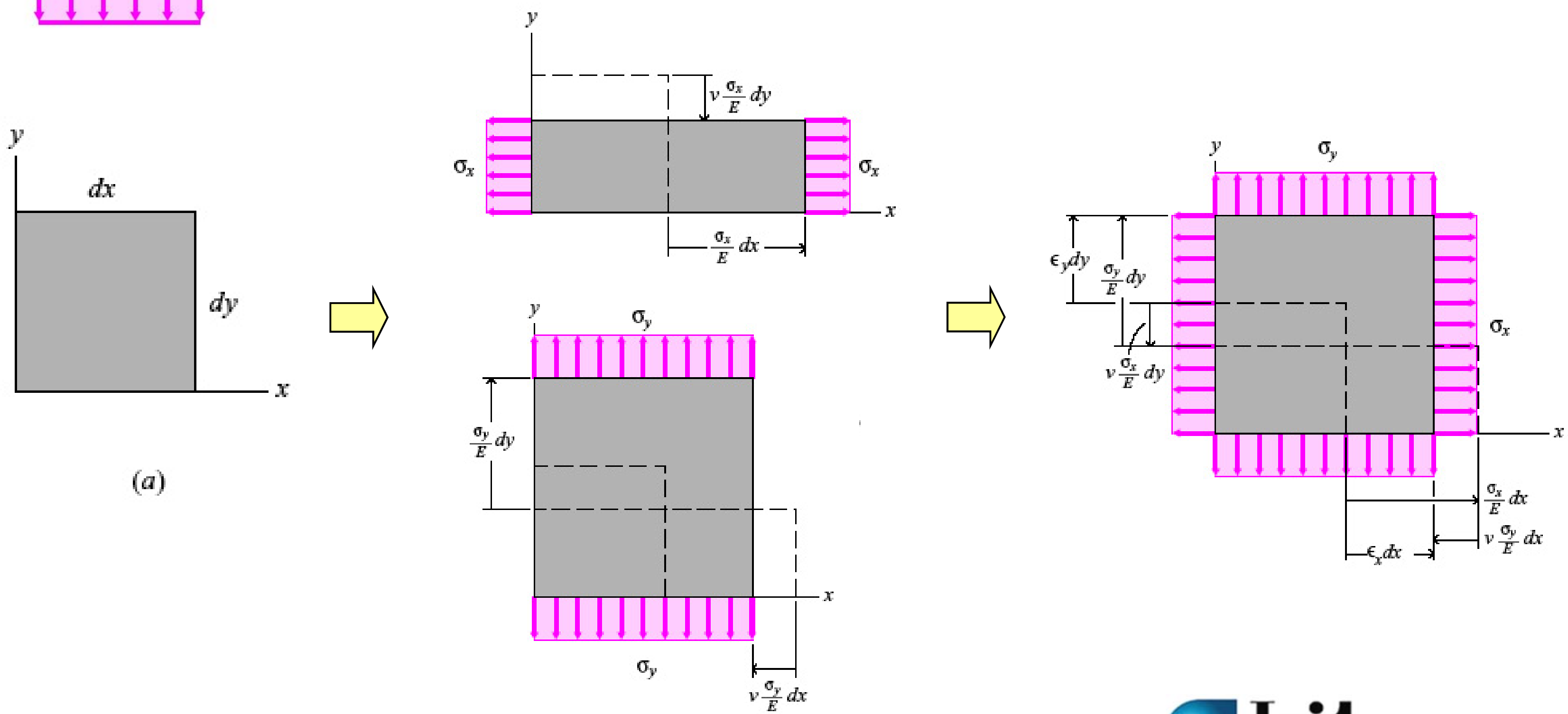
$$Modulus \ of \ rigidity : \quad G = \frac{E}{2(1+\nu)} = \frac{25{,}000}{2(1+0.35)} = 9{,}260 \ ksi$$

- Generalized Hooke's law

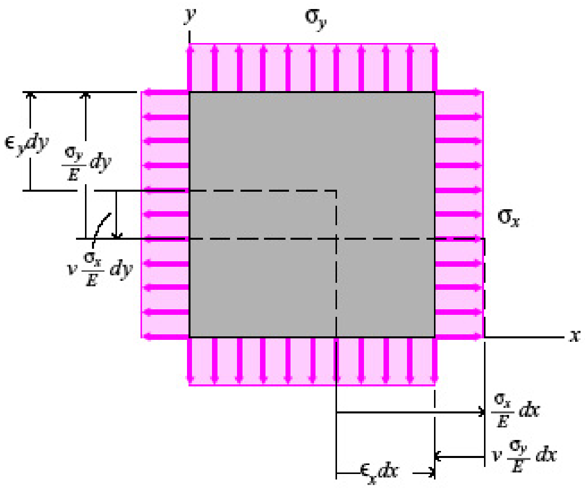


Principle of superposition

- If the stresses do not exceed the proportional limit and the deformations are small, the effects of separate loadings can be added algebraically



Stress-Strain Relations and Material Properties



$$d\delta_x = \varepsilon_x \, dx = \frac{\sigma_x}{E} dx - \nu \frac{\sigma_y}{E} dx$$

$$d\delta_y = \varepsilon_y \, dy = \frac{\sigma_y}{E} dy - \nu \frac{\sigma_x}{E} dy$$

$$\Rightarrow$$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right)$$

Extension to the state of tri-axial stresses

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\Rightarrow$$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\left(\varepsilon_y + \varepsilon_z\right)\right]$$

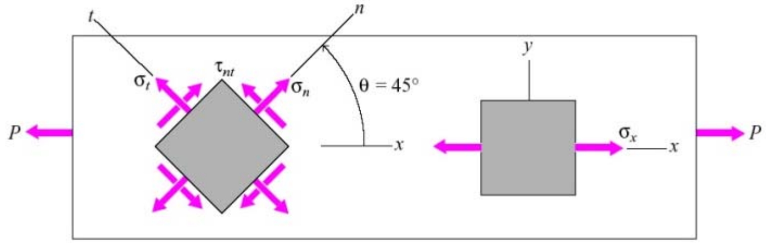
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\left(\varepsilon_z + \varepsilon_x\right)\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\varepsilon_x + \varepsilon_y\right)\right]$$

For the case of plane stress

$$\sigma_z = 0 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu\left(\varepsilon_x + \varepsilon_y\right)\right] \Rightarrow \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

Stress-Strain Relations and Material Properties



$$\tau_{nt} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -(\sigma_x - 0)\sin(+45)\cos(+45) + 0$$
$$= -\frac{\sigma_x}{2}$$

$$\gamma_{nt} = -2(\varepsilon_x - \varepsilon_y)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= -2(\varepsilon_x - \varepsilon_y)\sin(+45)\cos(+45) + 0$$
$$= -(\varepsilon_x - \varepsilon_y)$$
$$= -(\varepsilon_x - (-\nu\varepsilon_x))$$
$$= -\varepsilon_x(1+\nu) \quad \Leftarrow \quad \nu = -\frac{\varepsilon_t}{\varepsilon_a} = -\frac{\varepsilon_y}{\varepsilon_x}$$
$$\varepsilon_y = -\nu\varepsilon_x$$

$$\tau_{nt} = G\gamma_{nt} = -G\varepsilon_x(1+\nu)$$
$$-\frac{\sigma_x}{2} = -G\left(\frac{\sigma_x}{E}\right)(1+\nu) \quad \Rightarrow \quad \boxed{G = \frac{E}{2(1+\nu)}}$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$
$$\tau_{yz} = G\gamma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$
$$\tau_{zx} = G\gamma_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx}$$

(Example 4-2) When the machine part of alloy steel with E=210GPa and ν=0.3 was subjected to a biaxial stress, the measured strains were ε_x=+1394μm/m, ε_y=-660μm/m, and γ_{xy}=2054μm/m. Determine (a) stress components at the point (b) principal stresses and maximum stress at the point. Locate the planes on which these stresses act.

(Sol)

(a)

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) = \frac{210\times 10^9}{1-(0.3)^2}\left[1394 + (0.3)(-660)\right](10^{-6})$$

$$= +270\,(10^6)\,N/m^2 = 276\ MPa\ (T)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) = \frac{210\times 10^9}{1-(0.3)^2}\left[-660 + (0.3)(1394)\right](10^{-6})$$

$$= -55.8\,(10^6)\,N/m^2 = 55.8\ MPa\ (C)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{210\times 10^9}{2(1+0.3)}(2054)(10^{-6})$$

$$= 165.9\,(10^6)\,N/m^2 = 165.9\ MPa$$

(b)

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right) + \tau_{xy}^2}$$

$$= \frac{276 - 55.8}{2} \pm \sqrt{\left(\frac{276 + 55.8}{2}\right)^2 + (165.9)^2}$$

$$= 110.1 \pm 234.6$$

$$\sigma_{p1} = 344.7\ MPa = 345\ MPa\ (T)$$

$$\sigma_{p2} = -124.5\ MPa = 124.5\ MPa\ (C)$$

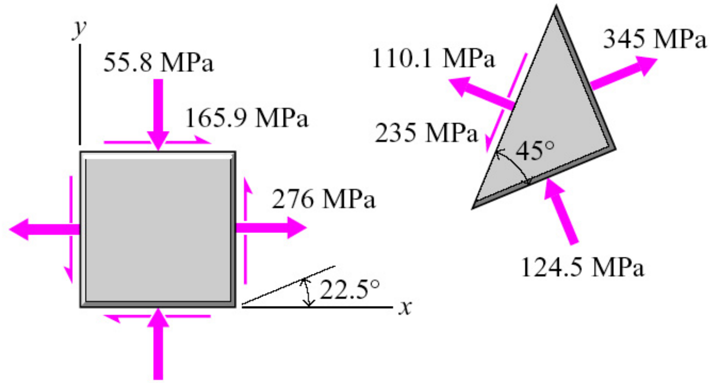
$$\sigma_{p3} = \sigma_z = 0$$

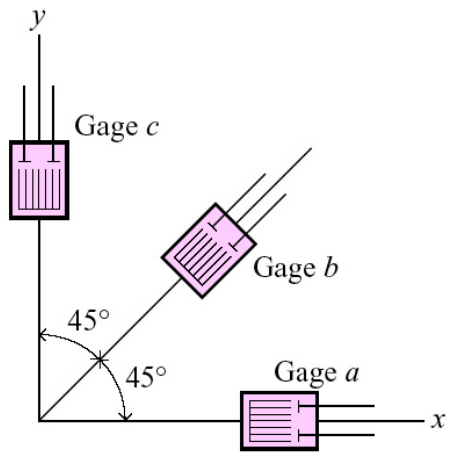
$$\tau_{\max} = \tau_p = \frac{1}{2}\left(\sigma_{p1} - \sigma_{p2}\right)$$

$$= \frac{1}{2}\left(344.7 - (-124.5)\right) = 234.6\ MPa$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(165.9)}{276.5 - (-55.8)} = 1.0$$

$$\Rightarrow \quad \theta_p = 22.5$$





(Example 4-3) When the machine part of alloy steel with E=30,000ksi and ν=0.3 was subjected to a biaxial stress, the measured strains were ε_a=+650μin/in, ε_b=+475μin/in, and ε_c=-250μin/in. Determine (a) stress components at the point (b) principal strains and maximum strain at the point. Locate the planes on which these stresses act (c) principal stresses and maximum stress at the point. Locate the planes on which these stresses act.

(Sol) (a)

$$\varepsilon_a = \varepsilon_x = +650\,\mu \qquad \varepsilon_b = +475\,\mu \qquad \varepsilon_c = \varepsilon_y = -250\,\mu$$

$$\varepsilon_b = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta = +475\,\mu \quad \Rightarrow \quad \gamma_{xy} = 550\,\mu$$

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) = \frac{30,000}{1-(0.3)^2}\left[650 + (0.3)(-250)\right](10^{-6}) = +18.96\ ksi = 18.96\ ksi\ (T)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) = \frac{30,000}{1-(0.3)^2}\left[-250 + (0.3)(650)\right](10^{-6}) = -1.813\ ksi = 1.813\ ksi\ (C)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{30,000}{2(1+0.3)}(550)(10^{-6}) = 6.35\ ksi = 6.35\ ksi$$

Stress-Strain Relations and Material Properties

(b)

$$\varepsilon_{p1}, \varepsilon_{p2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 200\,\mu \pm 527.4\,\mu$$

$$\varepsilon_{p1} = 200\,\mu + 527.4\,\mu \cong 727\,\mu$$

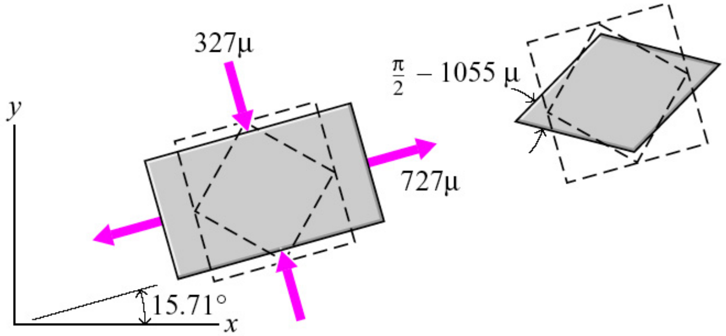
$$\varepsilon_{p2} = 200\,\mu - 527.4\,\mu \cong -327\,\mu$$

$$\varepsilon_{p3} = \varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) = -171.4\,\mu$$

$$\gamma_{\max} = \gamma_p = \varepsilon_{p1} - \varepsilon_{p2} = 1055\,\mu$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{550\,\mu}{650\,\mu - (-250\,\mu)} = 0.61111$$

$$2\theta_p = 31.43 \quad \Rightarrow \quad \theta_p = 15.71$$



(c)

$$\sigma_{p1} = \frac{E}{1-\nu^2}\left(\varepsilon_{p1} + \nu\varepsilon_{p2}\right)$$

$$= \frac{30{,}000}{1-0.3^2}\left[727.4 + 0.3(-327.4)\right](10)^6$$

$$= +20.742\ ksi \cong 20.7\ ksi\ \ (T)$$

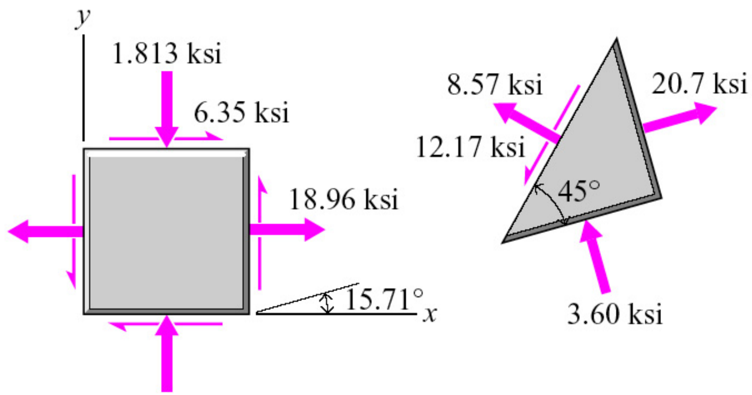
$$\sigma_{p2} = \frac{E}{1-\nu^2}\left(\varepsilon_{p2} + \nu\varepsilon_{p1}\right)$$

$$= \frac{30{,}000}{1-0.3^2}\left[-327.4 + 0.3(727.4)\right](10)^6$$

$$= -3.599\ ksi \cong 3.60\ ksi\ \ (C)$$

$$\sigma_{p3} = \sigma_z = 0$$

$$\tau_{\max} = \frac{E}{2(1+\nu)}\gamma_{\max} = \frac{30{,}000}{2(1+0.3)}(1054.8)(10)^6 = 12.17\ ksi$$

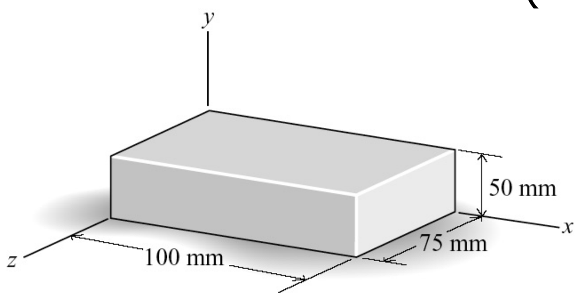


- Thermal strain
  - Most materials when unconstrained expand under heating and contract under cooling
  - Coefficient of thermal expansion : thermal strain due to one degree change in temperature

$$\varepsilon_T = \alpha \Delta T$$

- Total strains
  - Strains due to temperature change and strains due to applied loads are independent
  - Most materials expand uniformly in all directions when heated
  - Neither the shape of the body nor the shear stresses and shear strains are affected by temperature changes

$$\varepsilon_{total} = \varepsilon_\sigma + \varepsilon_T = \frac{\sigma}{E} + \alpha \Delta T$$



(Example) The aluminum [E=70GPa and α=22.5(10^{-6})/^oC] block rests on a smooth, horizontal surface. When the body is subjected to a temperature change of ΔT=20 ^oC. Determine

(a) The thermal strains $(\varepsilon_{Tx},\ \varepsilon_{Ty},\ \varepsilon_{Tz})$

(b) The deformations in the coordinate directions $(\delta_x,\ \delta_y,\ \delta_z)$

(c) The shearing strain (γ_{xy})

(Sol) (a) Thermal strain :

$$\varepsilon_T = \varepsilon_{Tx} = \varepsilon_{Ty} = \varepsilon_{Tz} = \alpha\,\Delta T = 22.5(10^{-6})(20)$$
$$= 450(10^{-6})m/m = 450\,\mu m/m$$

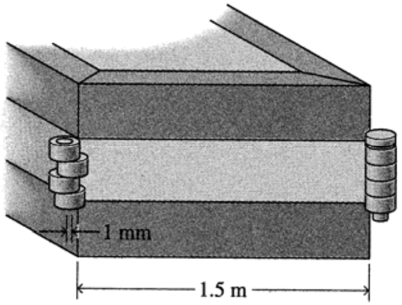
(b) Since the block is not constrained, there are no applied forces and there are no stresses. The load portion of the strain is zero, .

$$\varepsilon_{total} = \varepsilon_\sigma + \varepsilon_T = \varepsilon_T$$

$$\delta_x = \varepsilon_{Tx} L = 450(10^{-6})(100) = 0.0450\,mm$$
$$\delta_y = \varepsilon_{Ty} L = 450(10^{-6})(50) = 0.0225\,mm$$
$$\delta_z = \varepsilon_{Tz} L = 450(10^{-6})(75) = 0.0338\,mm$$

(c) Since the block is not constrained, there are no stresses and the total strains the same as the thermal strain. The thermal strain is the same in all directions. The original angle do not change. That is, there are no shearing strain, .

$$\gamma_{xy} = 0$$



(Example) Consider a 1.5m long brass [E=100GPa and α=17.6(10^{-6})/^{o}C] strap. Determine (a) how much the strap would have to be heated to lengthen the strap enough to insert the pin through the eyelets. (b) stress existing in the strap after the strap cooed back to room temperature.

(Sol) (a) Total strain : $\varepsilon = \frac{\delta}{L} = \frac{\sigma}{E} + \alpha \Delta T$

If no stress is applied to the strap and the temperature is raised to stretch the strap by 1mm (0.001m)

$$\delta = \left(\frac{\sigma}{E} + \alpha \Delta T \right) L = \left[0 + (17.6)(10^{-6}) \Delta T \right] (1.5) = 0.001$$

$$\Delta T = 37.9 \, ^{o}C$$

(b) If the strap stays stretched by 1mm after it cools down

$$\delta = \left(\frac{\sigma}{E} + \alpha \Delta T \right) L = \left[\frac{\sigma}{100\,(10^{9})} + 0 \right] (1.5) = 0.001$$

$$\sigma = 66.67(10^{6})\, N/m^{2} = 66.7\, MPa \;\; (T)$$

(Homework)

(4.5), (4.16), (4.19), (4.27), (4.31)