# Lecture Note for Solid Mechanics

## - Torsional Loading -

**Prof. Sung Ho Yoon**
**Department of Mechanical Engineering**
**Kumoh National Institute of Technology**

- **Text book : Mechanics of Materials, 6$^{th}$ ed.,**

  **W.F. Riley, L.D. Sturges, and D.H. Morris, 2007.**

- **Prerequisite : Knowledge of Statics, Basic Physics, Mathematics, etc.**

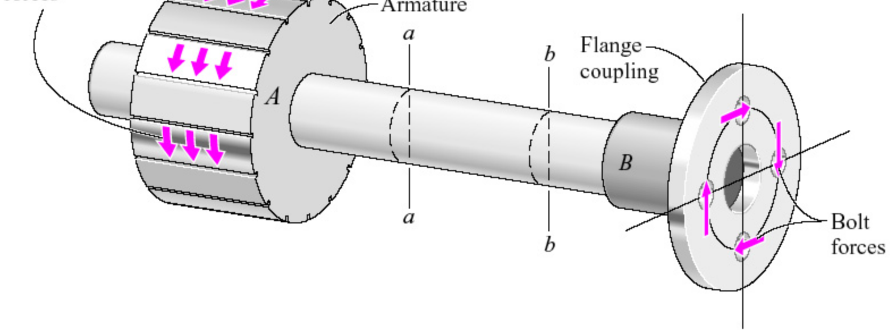
# Torsional Loading

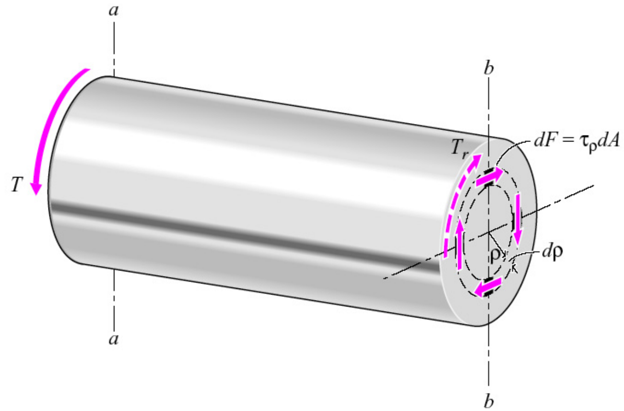
- Typical example of torsional loading problems :

  - circular shaft connecting an electric motor with other machine.

  - circular shaft transmits the torque from the armature to the coupling

  - determination of significant stresses and deformation of the circular shaft



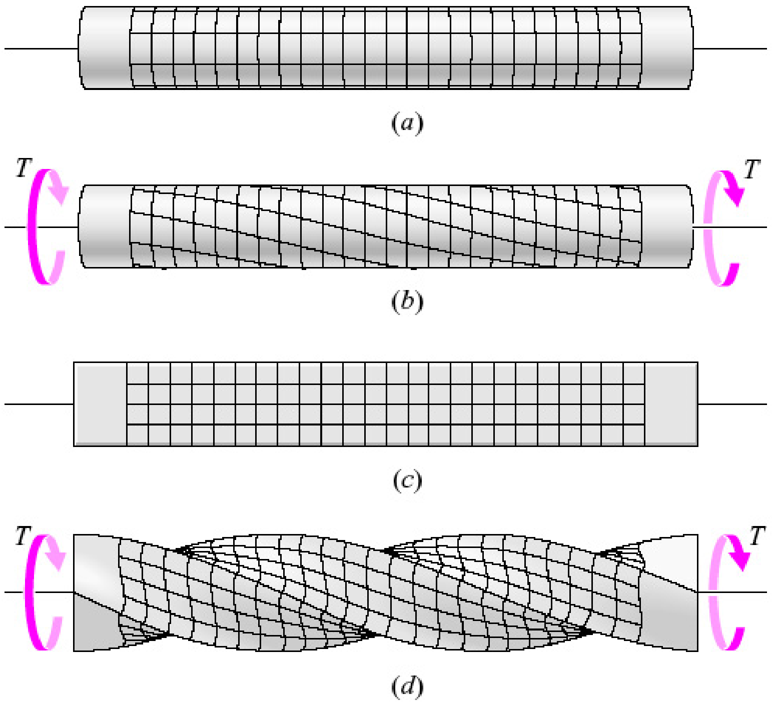




$$T = T_r = \int_{area} \rho dF = \int_{area} \rho \tau_\rho dA$$

# Torsional Loading

- Torsional loading problem :

  - Transverse cross section before twisting remains plane after twisting
  - Diameter of the section remains straight

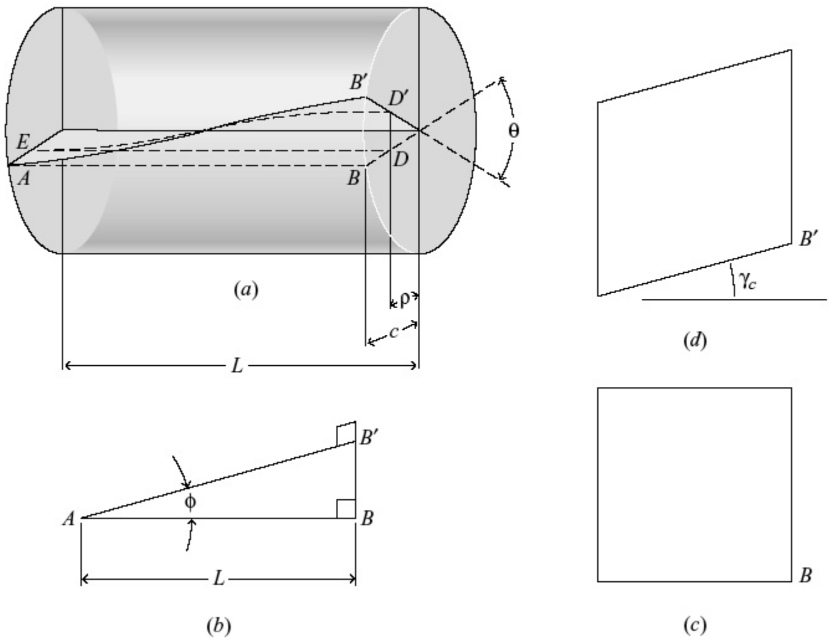



Plane sections remain plane

Plane sections become warped

# Torsional Loading

- Torsional shear strain




If the small strain is assumed

$$\tan\gamma_c \approx \gamma_c = \frac{BB'}{AB} = \frac{c\theta}{L}$$

$$\tan\gamma_\rho \approx \gamma_\rho = \frac{DD'}{ED} = \frac{\rho\theta}{L}$$

Combining above equations

$$\theta = \frac{\gamma_c L}{c} = \frac{\gamma_\rho L}{\rho}$$

Torsional shear strain becomes

$$\boxed{\gamma_\rho = \frac{\gamma_c \rho}{c}}$$

$\theta$ : *angle of twist*

$\gamma_\rho$ : *shear strain at a distance* $\rho$

$\gamma_c$ : *shear strain at a surface*

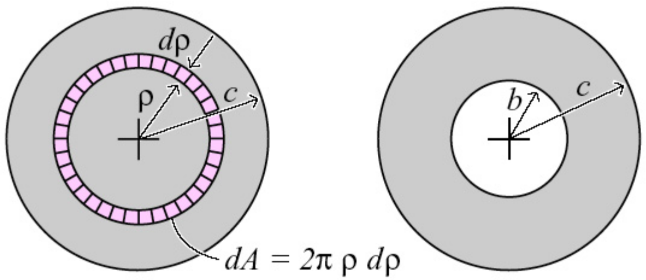
# Torsional Loading

If Hooke's law is assumed (stress is proportional to strain)

$$\gamma_\rho = \frac{\gamma_c \rho}{c} \quad \Rightarrow \quad \tau_\rho = \frac{\tau_c \rho}{c}$$

$$T = T_r = \frac{\tau_c}{c}\int \rho^2 dA = \frac{\tau_\rho}{\rho}\int \rho^2 dA \quad \Leftarrow \quad T = \int_A \rho \tau_\rho dA$$

$$where\ J = \int \rho^2 dA$$ : polar second moment of area




(a) (b)

For case (a) :

$$J = \int \rho^2 dA = \int_0^c \rho^2 (2\pi \rho\, d\rho) = \frac{\pi c^4}{2}$$

For case (b) :

$$J = \int \rho^2 dA = \int_b^c \rho^2 (2\pi \rho\, d\rho)$$

$$= \frac{\pi c^4}{2} - \frac{\pi b^4}{2} = \frac{\pi}{2}(r_o^4 - r_i^4)$$

# Torsional Loading

$$T = T_r = \frac{\tau_c}{c}\int \rho^2 dA = \frac{\tau_\rho}{\rho}\int \rho^2 dA \quad \Rightarrow \quad T = T_r = \frac{\tau_c J}{c} = \frac{\tau_\rho J}{\rho}$$

Unknown shear stress becomes

$$\tau_\rho = \frac{T\rho}{J} \quad \text{and} \quad \tau_c = \frac{Tc}{J}$$

Therefore,

$$\gamma_\rho = \frac{\gamma_c \rho}{c} \quad \text{and} \quad \tau_\rho = \frac{T\rho}{J}$$

- Shear strain is zero at the center of the shaft and increases linearly with respect to the distance from the axis of the shaft.
- Shear stress is zero at the center of the shaft and increases linearly with respect to the distance from the axis of the shaft.

# Torsional Loading

Determination of angle of twist

$$\gamma_\rho = \rho \frac{\theta}{L} \quad \text{or} \quad \gamma_\rho = \rho \frac{d\theta}{dL}$$

$$\tau_\rho = \frac{T\rho}{J} \quad \text{or} \quad \tau_c = \frac{Tc}{J}$$

$$G = \frac{\tau}{\gamma}$$

$$\Rightarrow \quad \boxed{\theta = \frac{\gamma_\rho L}{\rho} = \frac{\tau_\rho L}{G\rho} = \frac{TL}{GJ}}$$

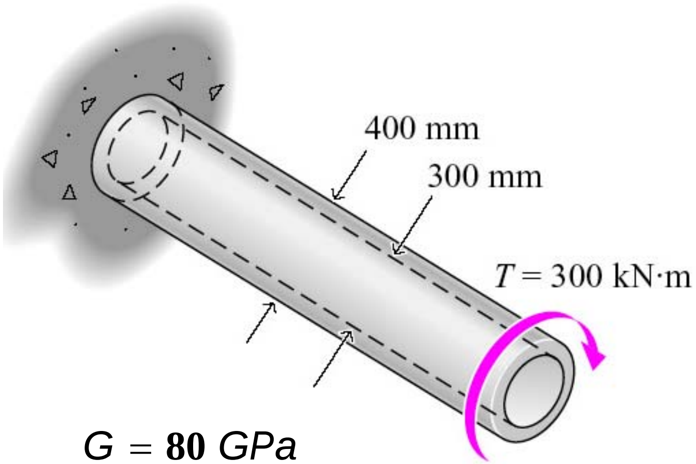
If T, G, or J is not constant along the length of the shaft

$$\theta = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$

If T, G, or J is a function of distance of the length of the shaft

$$\theta = \int_0^L \frac{T\,dx}{GJ}$$

# Torsional Loading




$G = 80\ GPa$

$$J = \frac{\pi}{2}(r_o^4 - r_i^4)$$

$$= \frac{\pi}{2}(200^4 - 150^4)$$

$$= 1718.1(10^6)\,mm^4$$

$$= 1718.1(10^{-6})\,m^4$$

(a) Maximum shear stress in the shaft

$$\tau_c = \tau_{max} = \frac{Tc}{J} = \frac{(300)(10^3)(200)(10^{-3})}{(1718.1)(10^{-6})}$$

$$= 34.92(10^6)\,N/m^2 \cong 34.9\,MPa$$

(b) Shear stress at the inside surface of the shaft

$$\tau_\rho = \frac{T\rho}{J} = \frac{(300)(10^3)(150)(10^{-3})}{(1718.1)(10^{-6})}$$

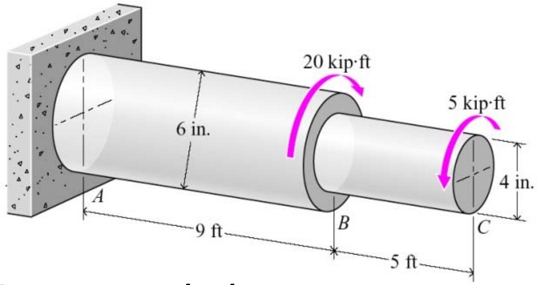
$$= 26.19(10^6)\,N/m^2 \cong 26.2\,MPa$$

(c) Magnitude of angle of twist in 2m length

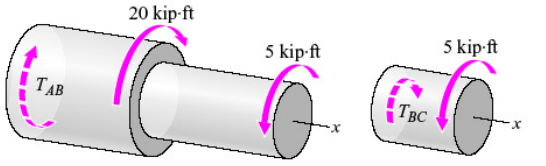
$$\theta = \frac{TL}{GJ} = \frac{(300)(10^3)(2)}{(80)(10^9)(1718.1)(10^{-6})}$$

$$= 0.004365\,rad \cong 0.00437\,rad$$

# Torsional Loading



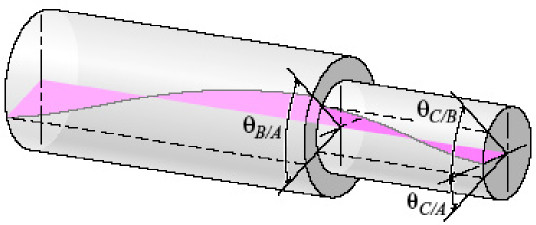

$G = 12{,}000\,ksi$ (a)




(b) (c)

$$J_4 = \frac{\pi}{2}c^4 = \frac{\pi}{2}(2^4) = 25.13\,in^4$$

$$J_6 = \frac{\pi}{2}c^4 = \frac{\pi}{2}(3^4) = 127.23\,in^4$$




(a) Maximum shear stress in the shaft

$$\tau_{AB} = \frac{T_{AB}c_{AB}}{J_{AB}} = \frac{(15)(12)(3)}{(127.23)} = 4.244\,ksi$$

$$\tau_{BC} = \frac{T_{BC}c_{BC}}{J_{BC}} = \frac{(5)(12)(2)}{(25.13)} = 4.775\,ksi \qquad \text{(max)}$$

(b) Rotation of end B w.r.t. end A

$$\theta_{B/A} = \frac{T_{AB}L_{AB}}{G_{AB}J_{AB}} = \frac{(15)(12)(9)(12)}{(12000)(127.23)} = 0.01273\,rad$$

(c) Rotation of end C w.r.t. end B

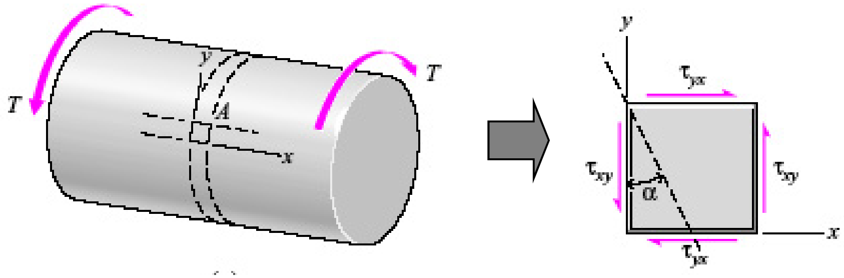
$$\theta_{C/B} = \frac{T_{BC}L_{BC}}{G_{BC}J_{BC}} = \frac{(5)(12)(5)(12)}{(12000)(25.13)} = 0.01194\,rad$$

(d) Rotation of end C w.r.t. end A

$$\theta_{C/A} = \theta_{B/A} - \theta_{C/B} = 0.01273 - 0.01194$$
$$= 0.000795\,rad$$

Questions: (1) transverse plane is a plane of maximum shearing stress?

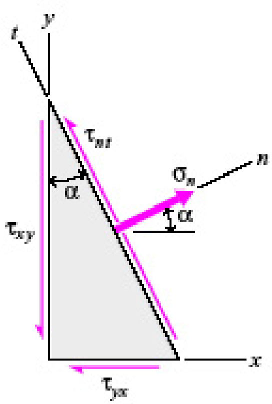
(2) there are other significant stresses induced by torsion?




From moment equilibrium equation:

$$\tau_{yx} = \tau_{xy}$$

From stress transformation equations:



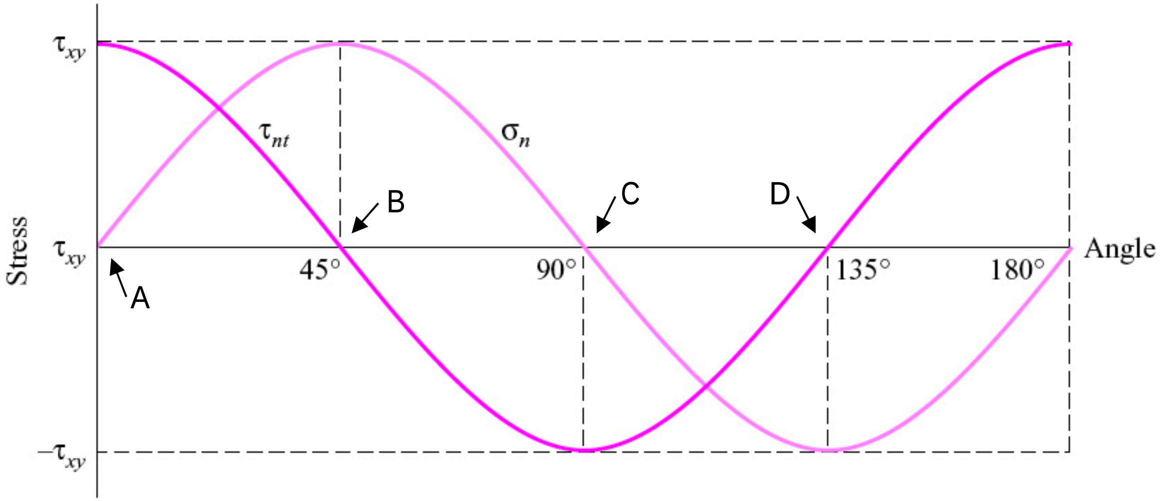

$$\begin{aligned}
\sigma_n &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\
&= \mathbf{0} + \mathbf{0} + 2\tau_{xy}\sin\alpha\cos\alpha = 2\tau_{xy}\sin\alpha\cos\alpha \\
&= \tau_{xy}\sin 2\alpha
\end{aligned}$$

$$\begin{aligned}
\tau_{nt} &= -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \\
&= \mathbf{0} + \tau_{xy}(\cos^2\alpha - \sin^2\alpha) = \tau_{xy}(\cos^2\alpha - \sin^2\alpha) \\
&= \tau_{xy}\cos 2\alpha
\end{aligned}$$

# Torsional Loading

Stresses on an inclined plane are functions of angle of inclined plane.

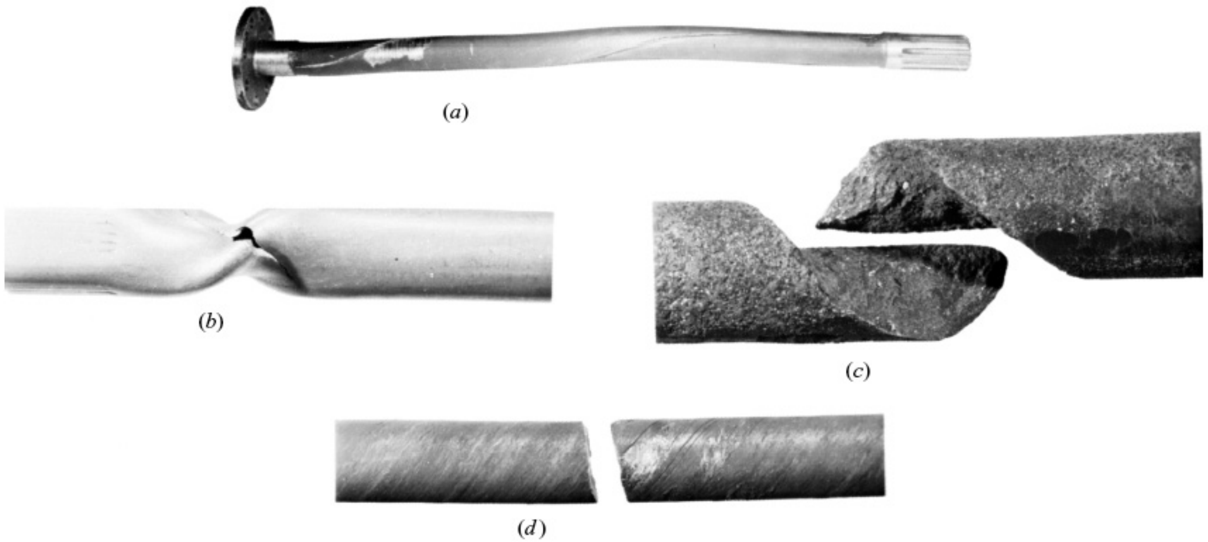
$$\sigma_n = \tau_{xy} \sin 2\alpha$$
$$\tau_{nt} = \tau_{xy} \cos 2\alpha$$




A & C : maximum shear stresses
B : maximum tensile stress
D : maximum compressive stress

# Torsional Loading

Type of failures



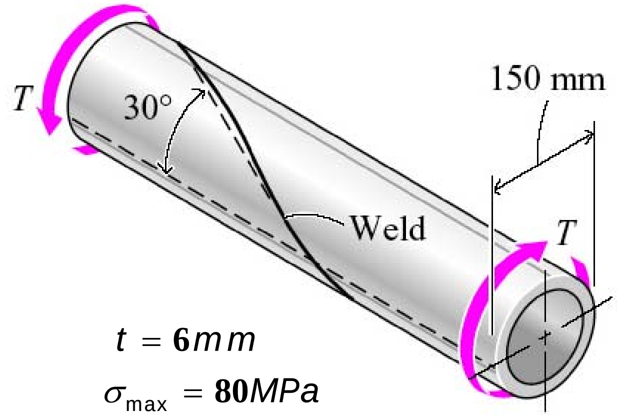

(a) Steel rear axle of a truck ⟹ Split longitudinally due to longitudinally running grains

(b) Thin-walled aluminum tube ⟹ Buckled and teared along 45 deg planes

(c) Gray cast iron of brittle material ⟹ Tensile failure

(d) Low carbon steel of ductile material ⟹ Shear failure

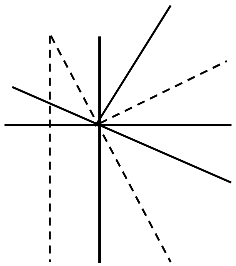
# Torsional Loading




$t = 6mm$

$\sigma_{max} = 80MPa$

$$J = \frac{\pi}{2}(75^4 - 69^4)$$
$$= 14.096\,(10^6)\,mm^4$$
$$= 14.096\,(10^{-6})\,m^4$$



(a) Maximum torque $T_{max}$ :

$$\sigma_{max} = \tau_{max} = \frac{T_{max}c}{J} = 80MPa = 80\,(10^6)\,N/m^2$$

$$T_{max} = \frac{\sigma_{max}J}{c} = \frac{(80)(10^6)(14.096)(10^{-6})}{(75)(10^{-3})}$$
$$= 15.04\,(10^3)\,N\cdot m = 15.04\,kN\cdot m$$

(b) FS when failure occurs at T=12kN-m, if ultimate strengths are 205MPa in shear and 345MPa in tension.

$$\sigma_n = \tau_{xy}\sin 2\alpha = \frac{Tc}{J}\sin 2\alpha$$
$$= \frac{12(10^3)(75)(10^{-3})}{(14.096)(10^{-6})}\sin 2(60^o) = 55.29\,MPa$$

$$\tau_{nt} = \tau_{xy}\cos 2\alpha = \frac{Tc}{J}\cos 2\alpha = -31.92MPa$$

$$\boxed{FS_\sigma = \frac{\sigma_{ult}}{\sigma_n} = \frac{345}{55.29} = 6.24}$$ and $$FS_\tau = \frac{\tau_{ult}}{\tau_{nt}} = \frac{205}{31.92} = 6.42$$

overall FS

# Torsional Loading

- Work done by a constant torque : $W_k = T\phi$

(where $\phi$ is angular displacement in radian)

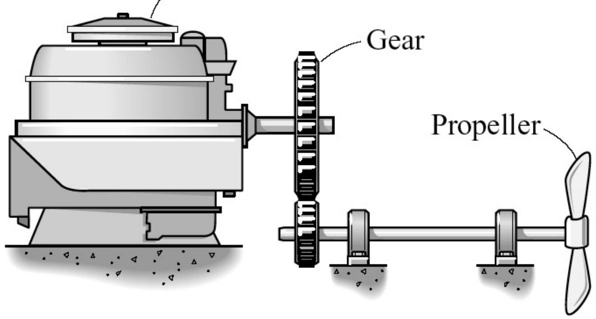
- Power is the time rate of doing work

$$Power = \frac{dW_k}{dt} = T\frac{d\phi}{dt} = T\omega$$

(where $\omega$ is angular velocity in rad/min)

(Example 6-8) Determine minimum permissible diameters of two shafts if allowable shearing stress is 20ksi and angle of twist in 10-ft of propeller shaft is not exceed 4-deg.

1HP=550ft-lb/s




*engine with* $200$*rpm and* $800$*hp*
*ratio gearbox to propeller* $= 4:1$

$$Power = T\omega \Longrightarrow 800(33{,}000) = T_1(200)(2\pi)$$

$$T_1 = 21{,}010\ ft\cdot lb\ \ for\ crank\ shaft$$

Assume no power loss,

$$T_2 = 5{,}252\ ft\cdot lb\ \ for\ propeller\ shaft$$

For crank shaft,

$$\frac{J}{c} = \frac{T}{\tau} \Leftrightarrow \frac{(\pi/2)\,c_1^4}{c_1} = \frac{(21{,}010)(12)}{(20)(10^3)} \Longrightarrow c_1 = 2.002\,in$$

$$d_1 = 4.004\,in$$

For propeller shaft,

$$\frac{J}{c} = \frac{T}{\tau} \Leftrightarrow \frac{(\pi/2)\, c_2^4}{c_2} = \frac{(5{,}252)(12)}{(20)(10^3)} \Longrightarrow c_2 = 1.261\, in$$

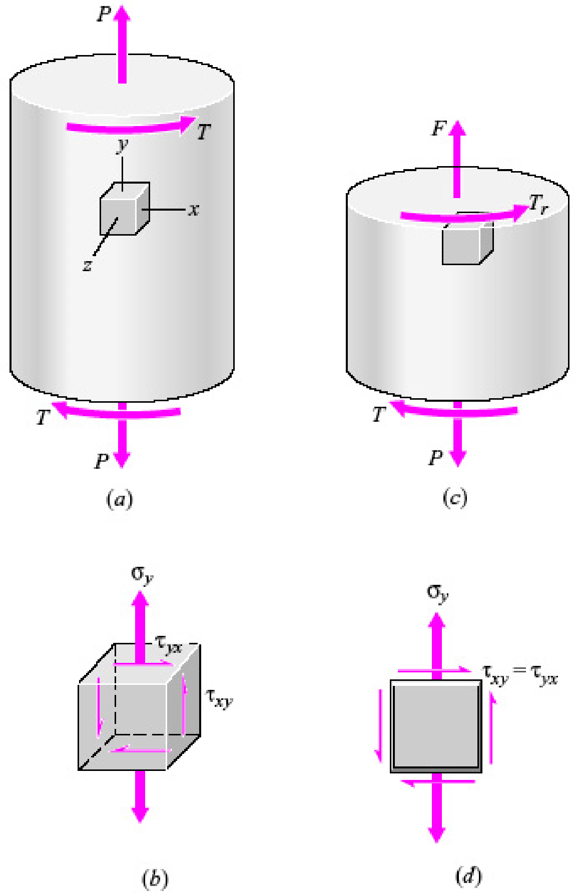
$$d_2 = 2.522\, in$$

$$\theta = \frac{TL}{GJ} \Leftrightarrow 4\frac{\pi}{180} = \frac{(5{,}252)(12)(10)(12)}{(12)(10^6)(\pi c_2^4/2)}$$

$$\Leftrightarrow c_2 = 1.5483 > 1.2612$$

Therefore, the propeller shaft should be diameter of 3.10-in.

# Torsional Loading




- Circular bar subjected to axial load P and torque T

$$\sigma = \frac{P}{A} \quad \text{due to axial load P}$$

$$\tau = \frac{T\rho}{J} \quad \text{due to torque T}$$

- As long as the strains are small, these stresses can be computed separately and superimposed on the element

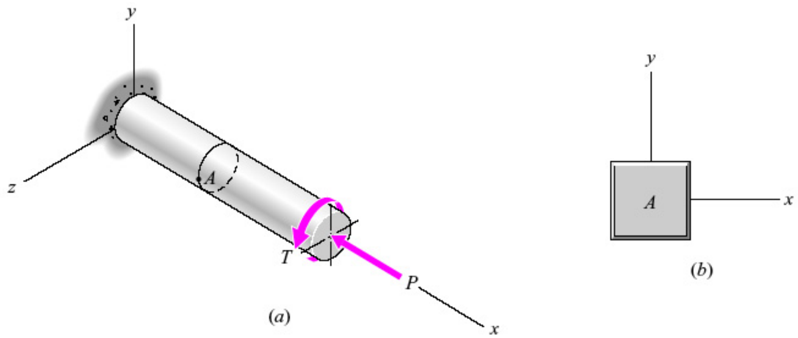
- Shearing stresses exist on both transverse and longitudinal planes

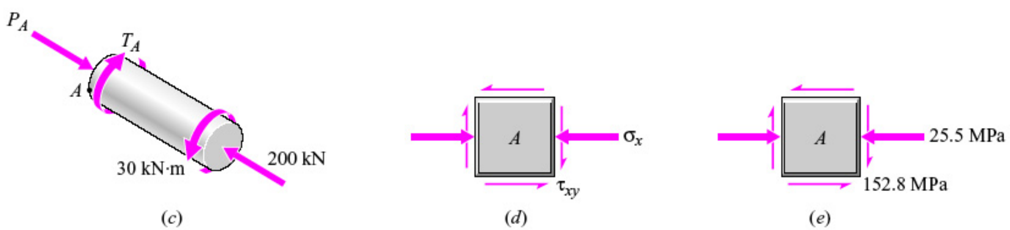
$$\tau_{xy} = \tau_{yx}$$

- Once stresses on the planes in (d) are known, stresses on any other plane through the point, as well as principal stresses and maximum shearing stress at the point can be found.

# Torsional Loading

(Example 6-12) Determine (a) x- and y-components of stresses, (b) principal stresses and maximum shearing stress at the point for point A on outside surface.




(Sol)

(a) $$\sigma_x = \frac{P_A}{A} = \frac{200(10^3)}{(\pi/4)(0.1)^2} = 25.46\,MPa\ (C)$$

$$\tau_{xy} = \frac{T_A c}{J} = \frac{30(10^3)(0.05)}{(\pi/2)(0.05)^4} = 152.79 MPa$$

(b) $$(\sigma_x,\ \sigma_y,\ \tau_{xy}) = (-25.46,\ 0,\ -152.79)$$

$$\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-25.46 + 0}{2} \pm \sqrt{\left(\frac{-25.46 - 0}{2}\right)^2 + (-152.79)^2}$$

$$= -12.73 \pm 153.32$$

Principal stresses and maximum shearing stress are

$$\sigma_{p1} = -12.73 + 153.32 = 140.59\ MPa\ (T)$$

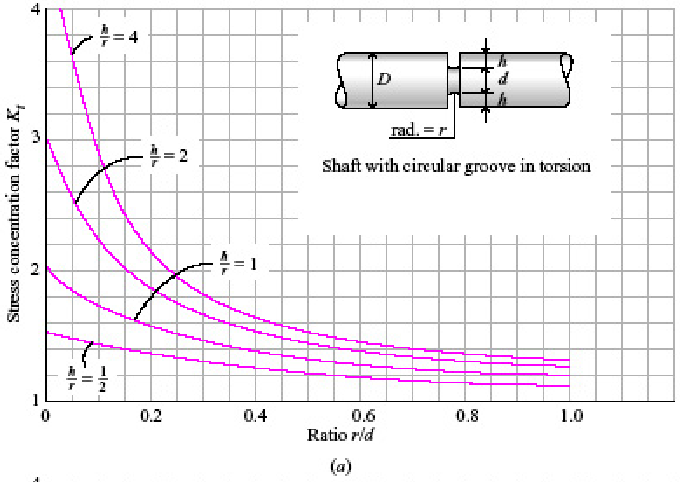
$$\sigma_{p2} = -12.73 - 153.32 = -166.05\ MPa\ (C)$$

$$\sigma_{p3} = \sigma_z = 0$$

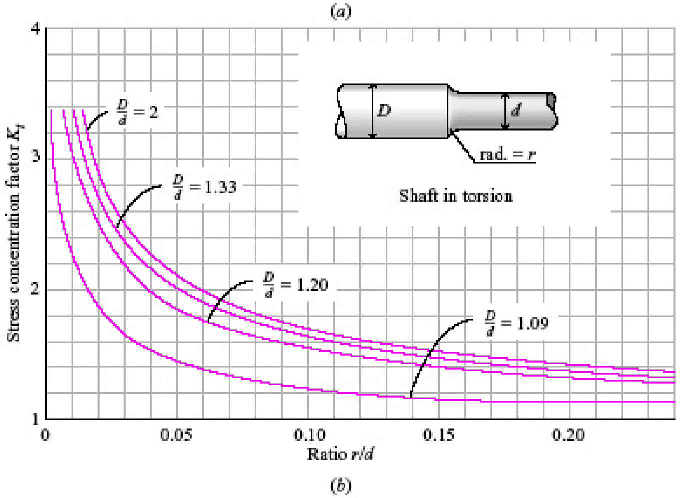
$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

$$= \frac{140.59 - (-166.05)}{2} = 153.32\ MPa$$

# Torsional Loading




(a)




(b)

$K_t$ : based on net section

- Circular bar subjected to torque T

$$\tau_{max} = \tau_c = \frac{Tc}{J}$$

- Stress concentrations occur in the vicinity of abrupt changes in diameter.
- Stress concentration factor K as

$$\tau_{max} = K\frac{Tc}{J} \qquad where \quad K = K\left(\frac{D}{d}, \frac{r}{d}\right)$$

(Example 6-14) For stepped shaft of D=4-in and d=2-in, determine min. fillet radius if max. shearing stress is 8ksi under torque of 6280 in-lb.

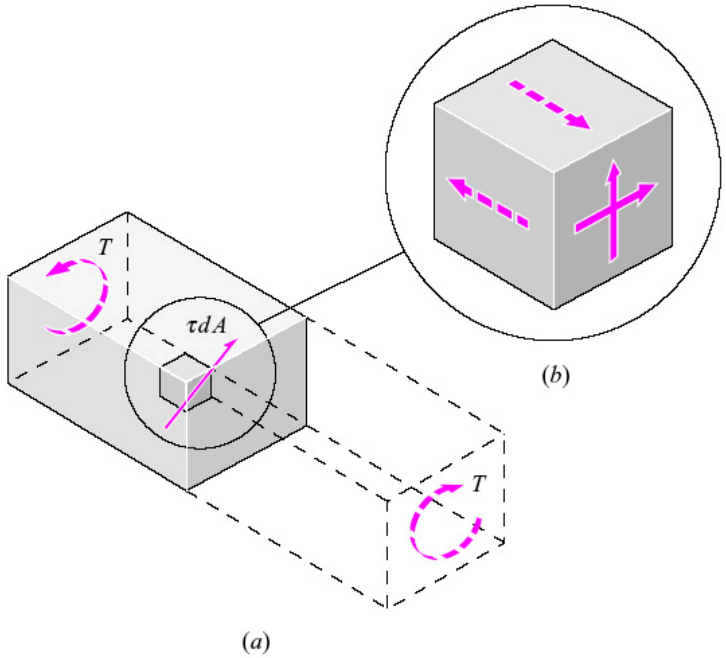
$$\tau_{max} = \frac{Tc}{J} = \frac{6280(1)}{(\pi/2)(1^4)} = 3998\ psi$$

$$K_t = \frac{8000}{3998} \approx 2.00$$

$$\frac{r}{d} = 0.06 \Rightarrow r = 0.06d = 0.12 - in$$

# Torsional Loading

- Shearing stresses in circular shaft are proportional to the distance from its axis.
- Shearing stresses for rectangular cross sections are not proportional to the distance from its axis.

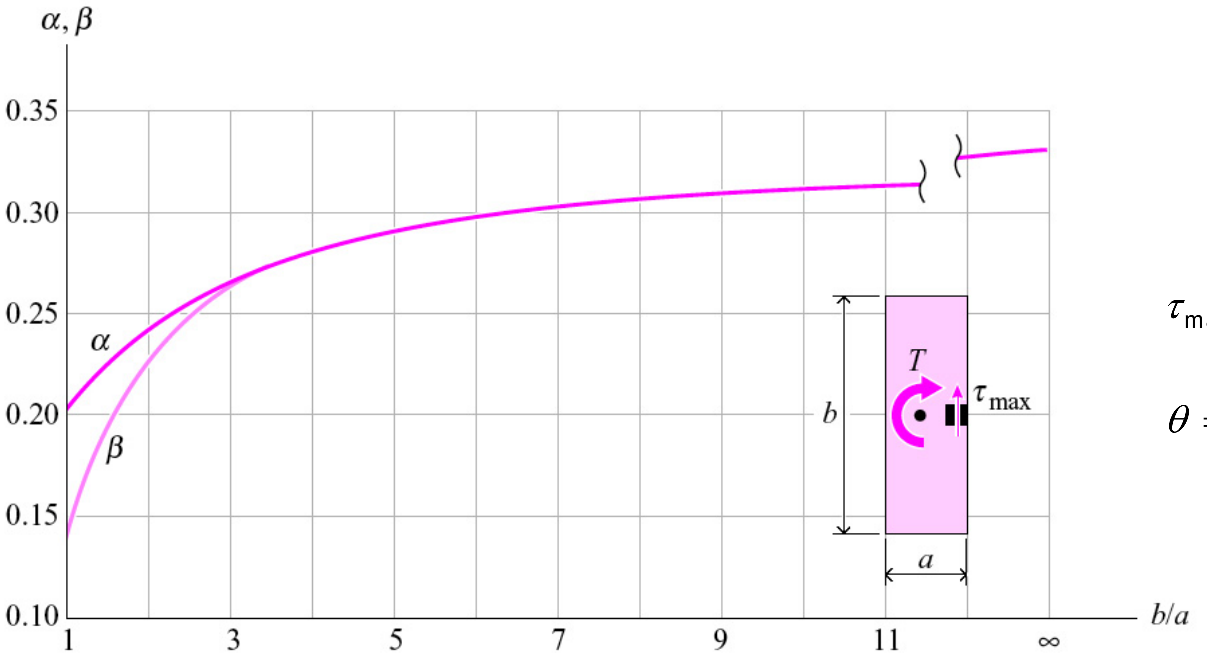



- Violate free boundary condition

- Shearing stresses at the corners of the rectangular bar must be zero.

# Torsional Loading

- Every cross sections, except for circular cross sections, will warp (not remain plane) when the bar is twisted.

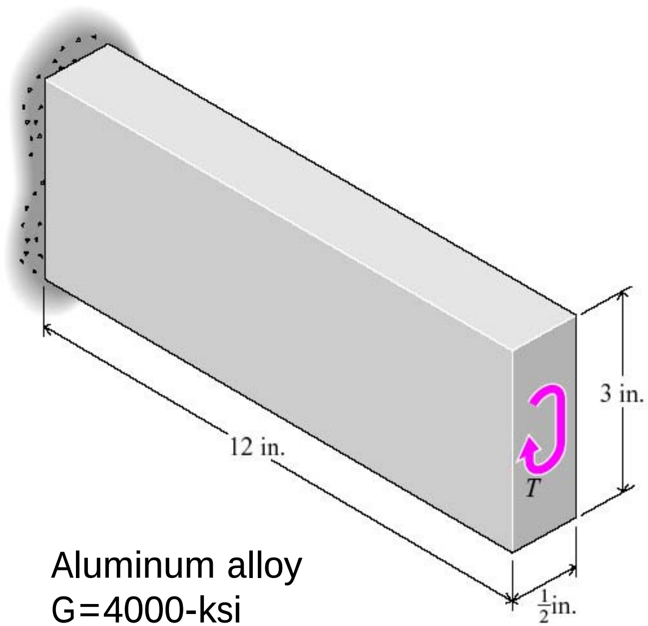



$$\tau_{max} = \frac{T}{\alpha a^2 b}$$

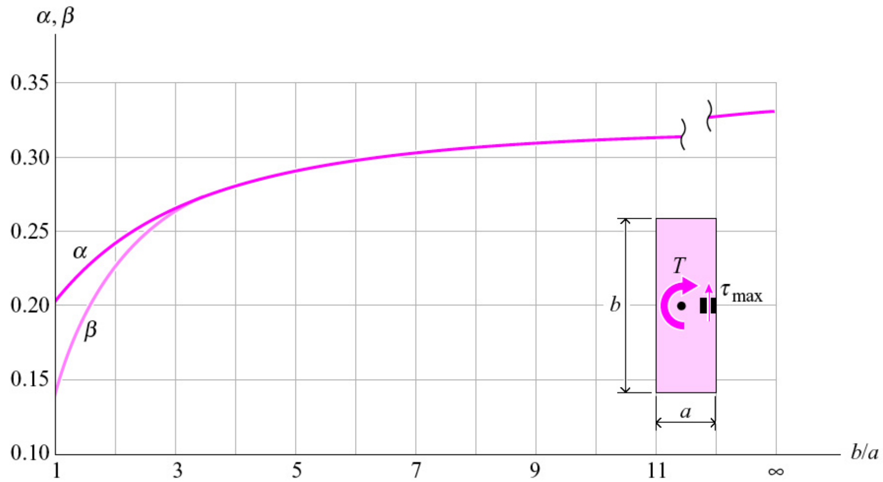
$$\theta = \frac{TL}{\beta a^3 bG}$$

# Torsional Loading

(Example 6-17) Determine maximum shearing stress and the angle of twist for 12-in length.




Aluminum alloy
G=4000-ksi
T=2500 in-lb



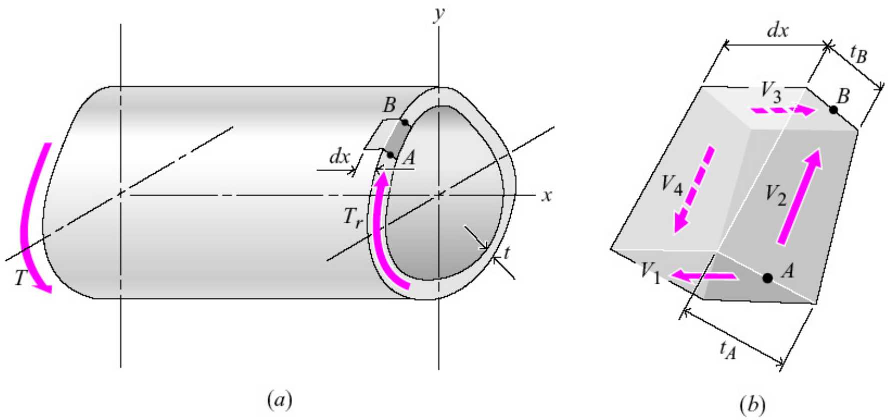

b/a=3/(1/2)=6 ⇨ $\alpha = \beta = 0.3$

$$\tau_{max} = \frac{T}{\alpha a^2 b} = \frac{2500}{0.3(1/2)^2(3)}$$
$$= 11{,}110\ psi$$

$$\theta = \frac{TL}{\beta a^3 bG} = \frac{2500(12)}{0.3(1/2)^3(3)(4000)(10^3)}$$
$$= 0.067\ rad$$

# Torsional Loading

- Elementary torsion theory : circular sections, thin-walled sections




$$\textbf{shear flow (q)} = \frac{\textbf{internal shear force}}{\textbf{unit length}}$$

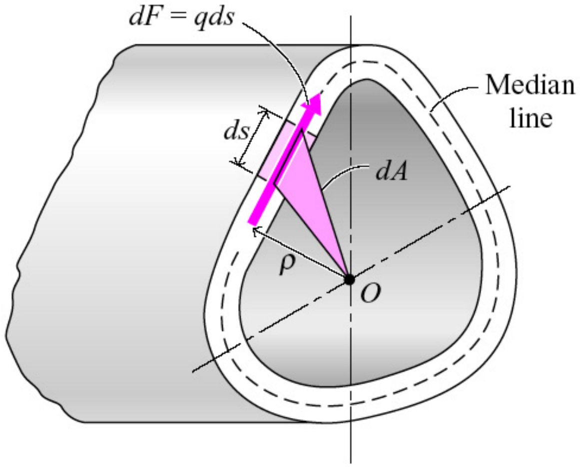
$$\boxed{q = \tau t}$$

where $\tau$ is average shearing stress across the thickness, t

- Shear flow on the cross section is constant even though the wall thickness varies.

$$\begin{aligned} V_1 = V_3 \quad &\Leftrightarrow \quad q_1 dx = q_3 dx \\ &\Leftrightarrow \quad q_1 = q_3 \\ &\Leftrightarrow \quad \tau_1 t_A = \tau_3 t_B \\ &\Leftrightarrow \quad \tau_A t_A = \tau_B t_B \\ &\Leftrightarrow \quad q_A = q_B \end{aligned}$$

# Torsional Loading



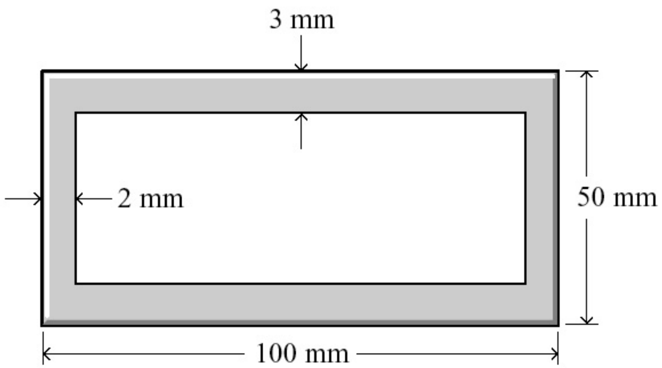

$$T_r = \int (dF)\rho = \int (qds)\rho = q\int \rho ds$$

$$T_r = 2Aq = 2A\tau t$$

$$\tau = \frac{T}{2At}$$

where A is the area enclosed by the median line.

(Example 6-18) Determine maximum torque that can be applied to the section if the maximum shearing stress must be limited to 95 MPa.




$$q = \tau t = 95\,(10^6)\,(2)\,(10^{-3}) = 190\,(10^3)\,N/m$$

$$T = 2Aq$$

$$= 2\,(100-2)\,(50-3)\,(10^{-6})\,(190)\,(10^3)$$

$$= 1750\,N-m$$

$$q = \tau t = 95(10^6)(3)(10^{-3}) = 285(10^3)\,N/m$$

$$T = 2Aq$$

$$= 2\,(100-2)(50-3)(10^{-6})(285)(10^3)$$

$$= 2625\,N-m$$

- Homework

(6-14), (6-47), (6-64), (6-76), (6-92), (6-126)