1.2 Laplace Transform and Inverse Transform

Laplace transform?

- Let f(t) be a given function which is defined for all $t \geq 0$
- We multiply f(t) by e^{-st} and integrate with respect to t from zero to infinity.
- Then, if the resulting integral exists, it is a function of S, say, F(s):

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

➡ F(s) : "Laplace Transform" of the function f(t) and will be denoted by $\mathscr{L}$(t)

Laplace transformation?

$$F(s) = \mathscr{L}(t) = \int_0^{\infty} e^{-st} f(t)dt \qquad (1)$$

➡ The operation just described, which yields F(s) from a given f(t), is called the "Laplace transformation", as distinguished from the transform.

Inverse Transform?

: The original function f(t) in eqn(1) is called the **"inverse transform"** or **"inverse of F(s)"** and will be denoted by $\mathcal{L}^{-1}(F)$: that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F)$$

<Notation>

Original functions are denoted by lower-case letters and their transforms by the same letters in capital, so that F(s) denotes the transform of f(t), and Y(s) denotes transform of y(t) and so on.

- (ex.1) Let f(t) = 1, when t≥0. Find F(s)
  - (sol.) From eqn(1)

$$\mathcal{L}(f) = \mathcal{L}(1) = \int_0^\infty e^{-st}dt = -\frac{1}{s} - e^{-st}\Big|_0^\infty;$$

Hence, when s>0, $\mathcal{L}(1) = \frac{1}{s}$

- (ex.2) Let f(t)=e^{at} when t≥0, where a is a constant.

  Find $\mathcal{L}$(f).

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{at}e^{-st}dt = \frac{1}{a-s}e^{-(s-a)t}\Big|_0^\infty \qquad ;$$

hence, when s−a>0, $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

(Question?)

"Must we go on in this fashion and obtain the transform of one function after another directly from the definition?"

➡ The Answer is "NO!"

: The reason is that the Laplace transformation has many general properties which are helpful for that purpose.

➡ A very important property : "Linearity"

(just as differentiation and integration)

<Theorem> (Linearity of the Laplace transformation)

The Laplace transformation is a linear operation, that is, for any function f(t) and g(t) whose Laplace transform exists and any constants a and b we have

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{f(t)\}$$

(proof) By the definition

$$\mathscr{L}\{af(t)+bg(t)\} = \int_0^\infty \mathrm{e}^{\text{-st}}\{af(t)+bg(t)\}dt$$

$$= a\int_0^\infty e^{-st} f(t)dt + b\int_0^\infty e^{-st} g(t)dt$$

$$= \mathrm{a}\mathscr{L}\{f(t)\} + \mathrm{b}\mathscr{L}\{g(t)\}$$

- (ex.3) $f(t) = \cosh at = \frac{1}{2}(e^{at} - e^{-at}) \quad \rightarrow \mathcal{L}(f)?$

  $\rightarrow a = jw \;\rightarrow\; f(t) = \cosh(jw) = \frac{1}{2}(e^{jwt} + e^{-jwt}) = \cos wt$

sol.) From Theorem(Linearity) and Example2, we obtain

$$\circ\ \mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\frac{2s}{s^2 - a^2}$$

$$\therefore \mathcal{L}(\cosh at) = \frac{s}{s^2 - a^2}$$

$$\circ\ \mathcal{L}(\sinh at) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)]$$

$$\therefore \mathcal{L}(\sinh at) = \frac{a}{s^2 - a^2}$$

Table1. some Elementary function f(t) and Laplace Transforms F(s) =$\mathcal{L}$(f)

	f(t)	$\mathcal{L}$(f)		f(t)	$\mathcal{L}$(f)
1	1	$\frac{1}{s}$	6	e^{at}	$\frac{1}{s-a}$
2	t	$\frac{1}{s^2}$	7	$\cos wt$	$\frac{s}{s^2+w^2}$
3	t^2	$\frac{2!}{s^3}$	8	$\sin wt$	$\frac{w}{s^2+w^2}$
4	t^n (n=1,2···)	$\frac{n!}{s^{n+1}}$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$

※ propertied of Gamma function : $\Gamma(\mathrm{a}+1)$

∘ Definition :

$$\Gamma(\mathrm{a}) \equiv \int_0^\infty e^{-t} t^{a-1} dt \qquad (\mathrm{a} > 0)$$

or $\Gamma(\mathrm{a}+1) \equiv \int_0^\infty e^{-t} t^{a} dt$

∘ Recurrence relation of the Gamma function.

- From the partial intergration method, we obtain

$$(fg)' = f'g + fg'$$

$$\therefore \int f'g = fg - \int fg'$$

$f = \int e^{-st} dt = -e^{-t}$

$g' = at^{a-1}$

$$\Gamma(a+1) = \int_0^\infty e^{-t} t^a dt \qquad (\mathrm{f}' = \mathrm{e}^{-t}, \mathrm{g} = \mathrm{t}^a)$$

$$= -\mathrm{e}^{t} t^a \Big|_0^\infty - \int_0^\infty (-e^{-t}) a t^{a-1} dt$$

$$= 0 + \mathrm{a}\int_0^\infty \mathrm{e}^{-t} t^{a-1} dt = \Gamma(a)$$

$$= \mathrm{a}\Gamma(\mathrm{a})$$

$$\therefore \Gamma(a+1) = a\Gamma(a)$$

* a = n : positive integer

$$\Gamma(1) = \int_0^\infty e^{-t}dt = -[e^{-t}]_0^\infty = -(0-1) = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(a) = 1!$$

$$\Gamma(3) = \Gamma(2+1) = 2\mathit{\Gamma}(2) = 2\cdot 1 = 2!$$

$$\vdots$$

$$\therefore \Gamma(\mathrm{n}+1) = \mathrm{n}!$$

<The proof of formulas in Table1>

- Proof of formulas 5 :

$$st = x \quad (dt = \frac{1}{s} dx)$$

$$\mathscr{L}(t^a) = \int_0^\infty e^{-st} t^a dt = \int_0^\infty e^{-x} (\frac{x}{s})^a \frac{1}{s} dx$$

$$= \frac{1}{s^{a+1}} \underbrace{\int_0^\infty e^{-x} x^a dx}_{\Gamma(a+1)} = \frac{\Gamma(a+1)}{s^{a+1}}$$

$$\therefore \mathscr{L}(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}$$

- proof of formula 4

From formula 5. where a = n(positive integer).

$$\mathscr{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

→ for the case of n=1 and n=2, formula 2 and 4 can be obtained directly

- proof of formulas 7 and 8

–First, Let us consider the Euler's formula

$$\begin{cases} e^{jwt} = \cos wt + j\sin wt \\ e^{-jwt} = \cos wt - j\sin wt \end{cases}$$

➡ usage in complex-number expressed in the polar form and in complex phasor notation.

$$\mathcal{L}(e^{jwt}) = \mathcal{L}(\cos\text{wt}) + \text{j}\mathcal{L}(\sin\text{wt})$$

$$\mathcal{L}(e^{jwt}) \underset{\uparrow}{=} \frac{1}{s - jw} = \frac{s + jw}{(s - jw)(s + jw)} = \frac{s}{s^2 + w^2} + j\frac{w}{s^2 + w^2}$$

(6)식

$$\therefore\ \mathcal{L}(\cos wt) = \frac{s}{s^2 + w^2}$$

$$\mathcal{L}(\sin wt) = \frac{w}{s^2 + w^2}$$