Set Theory

Sets and Subsets

- Set
  - A collection of objects
  - $A = \{a_1, a_2, a_3\}$
  - $a_2 \in A$: a_2 is an element of A.
  - $a_2 \notin A$: a_2 is not an element of A.
  - $a_i \in A,\ 1 \le i \le 3$.

  - Another representation

    $A = \{\ x \mid B(x)\ \}$, $B(x)$: x has blue eyes

    $A = \{\ x \mid x$ is an integer and $1 \le x \le 5\}$

    "the set of all x such that …"

- Notations of useful sets
  - $\mathbf{N} = \{0, 1, 2, \ldots\}$ set of natural numbers
  - $\mathbf{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ set of integers
  - $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ set of positive integers
  - $\mathbf{R}$: set of real numbers
  - $\mathbf{R}^+$: set of positive real numbers
  - $\mathbf{Q}$: set of rational numbers
  - $\varnothing$, $\{\}$: empty set or null set
  - U: universe or universe of discourse

- Cardinality (size) of a set
  - $|A|$: the number of elements in A (when it is *finite*).

- **Subset**
  - A set B is a subset of A ($B \subseteq A$) if every element of B is an element of A.
  - $B \subseteq A \;\Leftrightarrow\; (\forall x)(x \in B \Rightarrow x \in A)$
  - $B \subseteq A \;\Rightarrow\; |B| \leq |A|$

- **Theorem**

  For every set A, $A \subseteq U$, $A \subseteq A$, *and* $\varnothing \subseteq A$.

$$(\forall x)(x \in \varnothing \Rightarrow x \in A)$$

- **Set Equality**
  - A set A is equal to a set B iff $(A \subseteq B) \wedge (B \subseteq A)$.
  - $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$

- Proper Subset
  - If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B ($A \subset B$).
  - Note that $A \not\subset A$.
  - $B \subset A \;\Rightarrow\; |B| < |A|$
  - $B \subset A \;\Rightarrow\; B \subseteq A$

- Power Set
  - If A is a set then the power set of A, denoted $\wp(A)$, is the collection (or set) of all subsets of A.

$$\wp(A) = \{ B \mid B \subseteq A\}$$

(*Ex*) For a set $A = \{a, b\}$, $\wp(A) = \{ \varnothing, \{a\}, \{b\}, \{a, b\} \}$

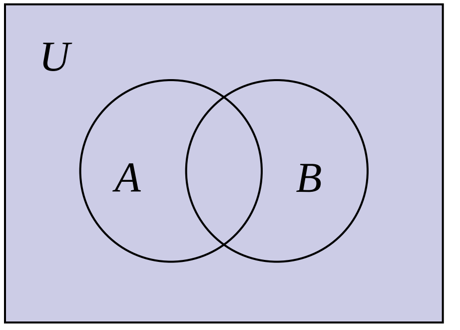
$|A| = 2$, $|\wp(A)| = 2^2$

- Theorem

In general, $|\wp(A)| = 2^{|A|}$.

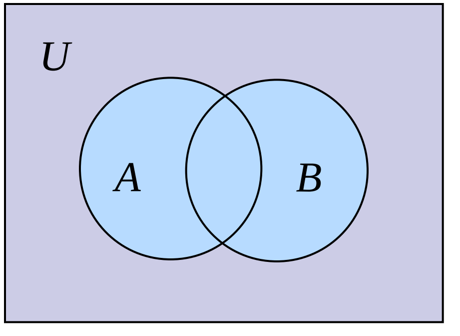
Set Operations

- Venn Diagram
  - Represents relations of sets
  - Does not constitute a proof



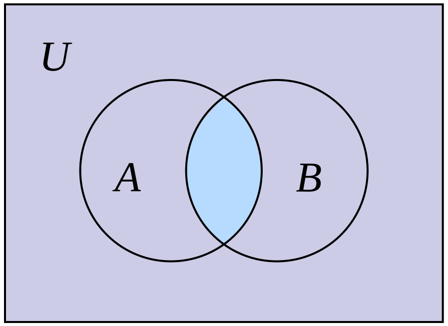
- Let A and B be two sets, then
  - $A \cup B = \{\, x \mid x \in A \vee x \in B \}$

    set union



- $A \cap B = \{ x \mid x \in A \wedge x \in B\}$

  set intersection

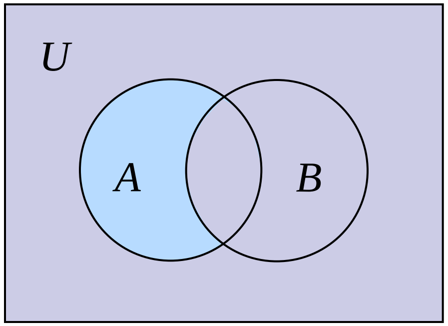


- $A - B = \{ x \mid x \in A \wedge x \notin B\}$

  set difference

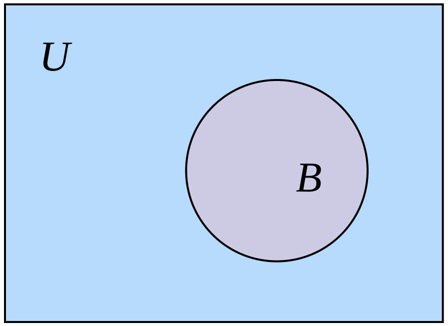
  (relative complement of B

  with respect to A)



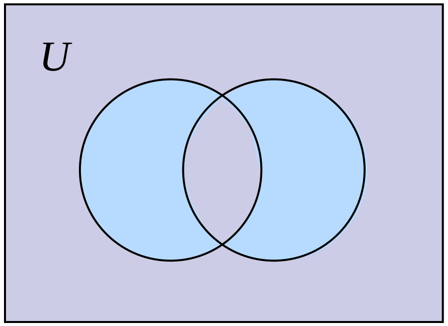
- $\overline{B} = U - B = \{ x \mid x \notin B \}$

  complement of a set B



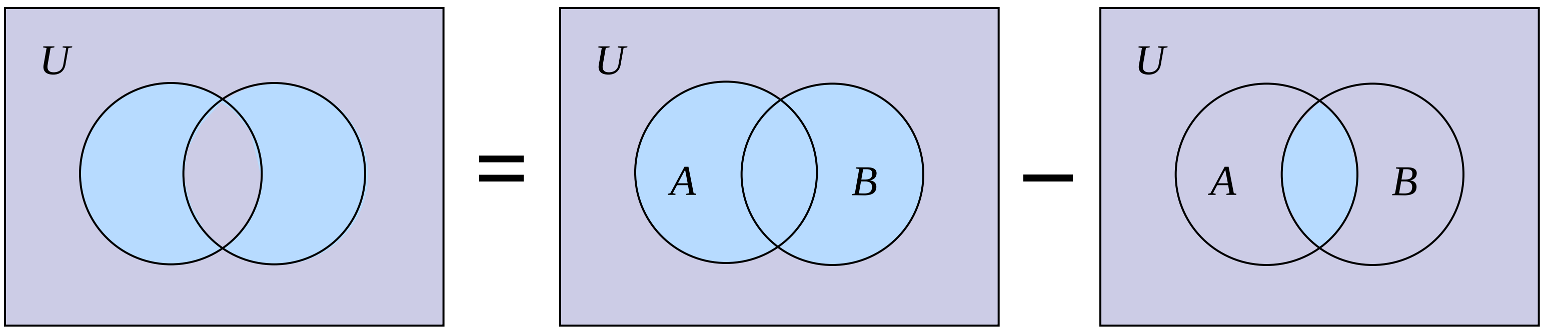
- $A \Delta B = \{ x \mid x \in A - B \vee x \in B - A \}$

  symmetric difference (of A and B)



- Theorem

$$A \,\Delta\, B = (A \cup B) - (A \cap B)$$



(*Note*)

$$A \,\Delta\, B = \{\, x \mid x \in A - B \vee x \in B - A \}$$

$$(A \cup B) - (A \cap B) = \{\, x \mid x \in A \cup B \wedge x \notin A \cap B \}$$

$$A - B = \{\, x \mid x \in A \wedge x \notin B \}$$

(*Proof*)

$$A \,\Delta\, B = \{\, x \mid x \in A - B \vee x \in B - A\}$$

$$= \{\, x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= \{\, x \mid ((x \in A \wedge x \notin B) \vee x \in B) \wedge ((x \in A \wedge x \notin B) \vee x \notin A)\}$$

$$= \{\, x \mid ((x \in A \vee x \in B) \wedge (x \notin B \vee x \in B))$$

$$\wedge ((x \in A \vee x \notin A) \wedge (x \notin B \vee x \notin A))\}$$

$$= \{\, x \mid (x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A)\}$$

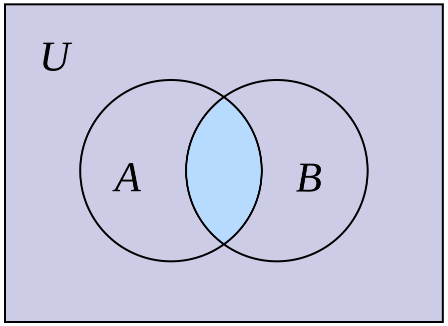
$$= \{\, x \mid x \in A \cup B \wedge (x \in \overline{B} \vee x \in \overline{A}\,)\}$$

$$= \{\, x \mid x \in A \cup B \wedge x \in \overline{A} \cup \overline{B}\ \}$$

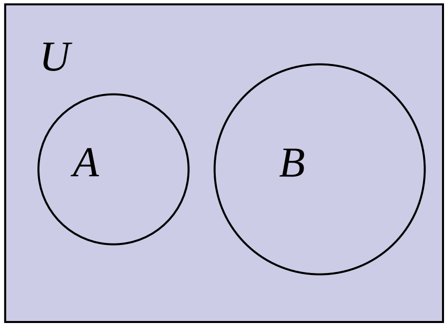
$$= \{\, x \mid x \in A \cup B \wedge x \in \overline{(A \cap B)}\}$$

$$= \{\, x \mid x \in A \cup B \wedge x \notin A \cap B\}$$

$$= (A \cup B) - (A \cap B)$$



- Disjoint Set
  - The sets A and B are said to be disjoint, or mutually disjoint if $A \cap B = \emptyset$.



- Theorem

  Let $A, B \subseteq U$.

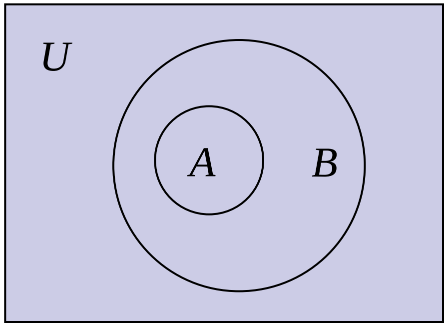
  A and B are disjoint iff $A \cup B = A \,\Delta\, B$.

- Theorem

  If $A, B \subseteq U$, then the following are equivalent:

  $a)\ A \subseteq B$ $\qquad b)\ A \cap B = A$

  $c)\ A \cup B = B$ $\qquad d)\ \overline{B} \subseteq \overline{A}$



(*Proof*)

To prove this theorem we need to show

$a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d$ (The proof consists of six parts).

Alternatively, however, we can just show the following:

$a \Rightarrow b, b \Rightarrow c, c \Rightarrow d, d \Rightarrow a$ (Only four parts)

<u>Part $(A \subset B) \Rightarrow (A \cap B = A)$</u>:

Assume $A \subseteq B$.

Let x be an arbitrary element of A, i.e., $x \in A$.

Then, $x \in B$ by the definition of subset.

Since $x \in A$ and $x \in B$, we know $x \in A \cap B$ by the definition of set intersection.

(*Proof*)

Since x is an arbitrary element of A, every element of A is an element of $A \cap B$. [UG]

Hence, by the definition of subset, $A \subseteq A \cap B$.

Let x be an arbitrary element of $A \cap B$, i.e., $x \in A \cap B$.

Then, by the definition of set intersection, $x \in A$ and $x \in B$.

Obviously, $x \in A$. [I_1]

Since x is an arbitrary element of $A \cap B$, every element of $A \cap B$ is an element of A. [UG]

Hence, by the definition of subset, $A \cap B \subseteq A$.

Therefore, $A \cap B = A$ by the definition of set equality.